Nilpotent Lie ring of order p^6

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1 Six generators

There is only one 6 generator nilpotent Lie ring of order p^6:

$$\langle a, b, c, d, e, f \mid \text{class } 1\rangle. \tag{6.1}$$

2 Five generators

Let L be a nilpotent Lie ring of order p^6, where L is generated by a, b, c, d, e, L/L_2 has order p^5, and L_2 has order p. If L is not abelian then L^2 has order p, and the commutator structure is determined by an alternating bilinear map on the øve dimensional vector space L/L^2 over $\mathbb{Z}_p$. So if L is not abelian the centre of L has order p^4 or p^2.

2.1 L abelian

If L is abelian we have

$$\langle a, b, c, d, e \mid ba, ca, da, ea, cb, db, eb, dc, ec, ed, pb, pc, pd, pe, \text{ class } 2\rangle. \tag{6.2}$$

2.2 The centre of L has order p^4

We may suppose that L^2 is generated by ba, and that all other Lie products of the generators (apart from ab) are zero. We may suppose that $\langle a, b\rangle$ is isomorphic to 3.2 or 3.3, so that we either have $pa = pb = 0$, or we have $pa = ba$, $pb = 0$. The centre of L is generated by c, d, e, ba. If the centre of L has characteristic p then we have

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pa, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{6.3}$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle. \tag{6.4}$$

If the centre of L does not have characteristic p then we may suppose that $pc = ba$, $pd = pe = 0$. Replacing a by $a - c$ if necessary, we may suppose that $pa = 0$. This gives

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pa, pb, pc - ba, pd, pe, \text{ class } 2\rangle. \qquad (6.5)$$

These three Lie rings are distinct, since 6.3 is the only one of characteristic p, and 6.5 is the only one in which the centre does not have characteristic p.

2.3 The centre of L has order p^2

We may suppose that $ba = dc$ and that

$$ca = da = ea = cb = db = eb = ec = ed = 0.$$

So $\langle a, b, c, d\rangle$ is isomorphic to 5.5 or 5.6, and e is central. Thus we have $pa = 0$ or ba, and $pb = pc = pd = 0$. We can also assume that $pe = 0$ or ba. However if $pe = ba$, then replacing a by $a - e$ if necessary we may assume that $pa = 0$. So we have three algebras:

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pa, pb, pc, pd, pe, \text{ class } 2\rangle, \qquad (6.6)$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle, \qquad (6.7)$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pa, pb, pc, pd, pe - ba, \text{ class } 2\rangle. \qquad (6.8)$$

These three Lie rings are distinct, since 6.6 is the only one of characteristic p, and 6.8 is the only one in which the centre does not have characteristic p.

3 Four generators

If L is a four generator nilpotent Lie ring of order p^6 then L must be an immediate descendant of 4.1, 5.2 or 5.3. (The algebras 5.4 ˘ 5.7 are all terminal.)

3.1 Descendants of 4.1

Let L be an immediate descendant of 4.1. Then L is generated by a, b, c, d, and L_2 has order p^2. So the derived algebra L^2 has order 1 or p or p^2.

3.1.1 L abelian

If L is abelian then we have

$$\langle a, b, c, d \mid ba, ca, da, cb, db, dc, pc, pd, \text{ class } 2\rangle. \qquad (6.9)$$

3.1.2 L^2 has order p

If L^2 has order p then we may assume that L^2 is generated by ba, and we may assume that

$$ca = da = cb = db = dc = 0,$$

or that

$$ca = da = cb = db = 0,\ dc = ba.$$

First suppose that $ca = da = cb = db = dc = 0$. Then the centre of L has order p^4 and equals $\langle c, d\rangle + L_2$.

If pc, pd are linearly independant then we may assume that $pc = ba$, and that L_2 is generated by ba, pd. Furthermore, subtracting suitable linear combinations of c, d from a, b we may assume that $pa = pb = 0$. So we have

$$\langle a, b, c, d \,|\, ca, da, cb, db, dc, pa, pb, pc - ba, \text{ class } 2\rangle. \tag{6.10}$$

Next suppose that pc, pd span a space of dimension at most 1. Then we may assume that $pd = 0$, so that the subalgebra $\langle a, b, c\rangle$ must have order p^5 and derived algebra of order p. So $\langle a, b, c\rangle$ is isomorphic to one of 5.9 ˘ 5.13. So we have

$$\langle a, b, c, d \,|\, ca,\ da,\ cb,\ db,\ dc,\ pb,\ pc,\ pd, \text{ class } 2\rangle, \tag{6.11}$$

$$\langle a, b, c, d \,|\, ca,\ da,\ cb,\ db,\ dc,\ pb - ba,\ pc,\ pd, \text{ class } 2\rangle, \tag{6.12}$$

$$\langle a, b, c, d \,|\, ca,\ da,\ cb,\ db,\ dc,\ pb,\ pc - ba,\ pd, \text{ class } 2\rangle, \tag{6.13}$$

$$\langle a, b, c, d \,|\, ca,\ da,\ cb,\ db,\ dc,\ pa,\ pb,\ pd, \text{ class} 2\rangle, \tag{6.14}$$

$$\langle a, b, c, d \,|\, ca,\ da,\ cb,\ db,\ dc,\ pa - ba,\ pb,\ pd, \text{ class } 2\rangle. \tag{6.15}$$

Note that in 6.11 and 6.12 the centre of L has characteristic p, whereas this is not the case in 6.13 ˘ 6.15. Furthermore pL has order p in 6.11 and 6.14, but order p^2 in 6.12, 6.13 and 6.15. Finally, 6.13 is not isomorphic to 6.15, since if we let C be the centre of L then $pC \leq L^2$ in 6.13, but not in 6.15. So the algebras 6.11 ˘ 6.15 are distinct.

Next suppose that $ca = da = cb = db = 0$, $dc = ba$ so that the centre of L is L_2. We must have that at least one of pa, pb, pc, pd is not a linear multiple of ba. So we may assume that pa is not a linear multiple of ba. Replacing b by $b - \alpha a$ for suitable α, we may suppose that $pb \in Sp\langle ba\rangle$. Similarly we may assume that $pd \in Sp\langle ba\rangle$. We would like to show that we can also assume that $pc \in Sp\langle ba\rangle$. So suppose for the moment that $pc \notin Sp\langle ba\rangle$. Then we can ønd $\beta \neq 0$ such that $p(c - \beta a) \in Sp\langle ba\rangle$. We let

$$\begin{aligned}
a' &= c + \beta a,\\
b' &= b + \beta d,\\
c' &= c - \beta a,\\
d' &= -b + \beta d.
\end{aligned}$$

It is straightforward to check that

$$\begin{aligned} b'a' &= 2\beta ba = d'c', \\ c'a' &= d'a' = c'b' = d'b' = 0, \end{aligned}$$

and that pb', pc', $pd' \in Sp\langle ba\rangle$. So we may assume that $pa \notin Sp\langle ba\rangle$, pb, pc, $pd \in Sp\langle ba\rangle$ as claimed. Scaling a and b we may assume that $pb = 0$ or ba. And similarly we may assume that $pc = 0$ or ba, $pd = 0$. This gives four algebras:

$$\langle a, b, c, d \mid ca,\ da,\ cb,\ db,\ dc - ba,\ pb,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.16}$$

$$\langle a, b, c, d \mid ca,\ da,\ cb,\ db,\ dc - ba,\ pb - ba,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.17}$$

$$\langle a, b, c, d \mid ca,\ da,\ cb,\ db,\ dc - ba,\ pb,\ pc - ba,\ pd,\ \text{class } 2\rangle, \tag{6.18}$$

$$\langle a, b, c, d \mid ca,\ da,\ cb,\ db,\ dc - ba,\ pb - ba,\ pc - ba,\ pd,\ \text{class } 2\rangle.$$

However the fourth of these algebras is isomorphic to 6.17. To see this we let $a' = a-d$, $b' = b$, $c' = -b + c$, $d' = d$ in the fourth algebra. It is easy to check that a', b', c', d' satisfy the relations of 6.17.

To show that 6.16 – 6.18 are all distinct we first note that pL has order p in 6.16, but order p^2 in 6.17 and 6.18. To show that 6.17 is not isomorphic to 6.18, we analyze all possible generating sets a', b', c', d' for L with

$$\begin{aligned} b'a' &= d'c', \\ c'a' &= d'a' = c'b' = d'b' = 0, \end{aligned}$$

and with $pa' \notin L^2$, pb', pc', $pd' \in L^2$. It is straightforward to show that b' must be a linear multiple of b modulo L_2. Since $pb \neq 0$ in 6.17, whereas $pb = 0$ in 6.18, it follows that these two algebras are distinct.

3.1.3 L^2 has order p^2

The commutator structure of L must be the same as that of a four generator 6 dimensional Lie algebra over $\mathbb{Z}_p$ with derived algebra of dimension 2. So we can assume that L^2 is spanned by ba, ca, and we can assume that one of the following four sets of relations are satisfied:

$$\begin{aligned} cb &= da = db = dc = 0, \\ cb &= da = dc = 0,\ db = ba, \\ cb &= da = dc = 0,\ db = ca, \\ cb &= da = 0,\ db = ca,\ dc = \omega ba. \end{aligned}$$

The four class 2 Lie algebras over $\mathbb{Z}_p$ defined by these four sets of relations are non-isomorphic. And it follows that if L and M are two Lie rings of class 2 and order p^6 satisfying different sets of commutator relations from this collection of four sets, then L cannot be isomorphic to M, whatever the power structures on L and M.

We consider each of these four sets of commutator relations in turn.

Case1 First suppose that $cb = da = db = dc = 0$. Note that d is central. The subalgebra $\langle a, b, c\rangle$ must be isomorphic to one of 5.14 ˘ 5.23, so if $pd = 0$ then we have

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.19}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa - ba,\ pb,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.20}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb - ba,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.21}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa - ca,\ pb - ba,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.22}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb - ca,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.23}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa - ba,\ pb - ca,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.24}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb - ba,\ pc - \lambda ca,\ pd,\ \text{class } 2\rangle \tag{6.25}$$

with $\lambda \neq 0$, where λ and λ^{-1} give isomorphic algegbras ($(p+1)/2$ algebras),

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb - ba - ca,\ pc - ca,\ pd,\ \text{class } 2\rangle, \tag{6.26}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb - \omega ca,\ pc - ba,\ pd,\ \text{class } 2\rangle, \tag{6.27}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb - \alpha ca,\ pc - ba - ca,\ pd,\ \text{class } 2\rangle \tag{6.28}$$

where $1 + 4\alpha$ is not a square ($(p-1)/2$ algebras). These algebras must be distinct, since if a', b', c' are any elements which span L modulo the centre of L, then $\langle a', b', c'\rangle$ is isomorphic to $\langle a, b, c\rangle$.

If $pd \neq 0$ then L cannot be isomorphic to any of 6.19 ˘ 6.28. Replacing a, b, c by $a - \lambda d$, $b - \mu d$, $c - \nu d$ for suitable λ, μ, ν we may assume that either $pa = pb = pc = 0$, or pa, pb, pc span a one dimensional subspace D, with $pd \notin D$. Clearly we may choose b, c so that $pd = ca$ and so that pa, pb, pc are all linear multiples of ba. If pb or pc is non-zero then, replacing a by $a - \beta b - \gamma c$ for suitable β, γ we may assume that $pa = 0$. So, by suitable scaling we may assume that $pd = ca$, and that one of the following sets of equations hold:

$$\begin{aligned}
pa &= pb = pc = 0, \\
pa &= ba,\ pb = pc = 0, \\
pa &= 0,\ pb = ba,\ pc = 0, \\
pa &= 0,\ pb = 0,\ pc = ba, \\
pa &= 0,\ pb = ba,\ pc = ba.
\end{aligned}$$

However the last two of these sets of equations give isomorphic algebras, as can be seen by assuming that the last set of equations holds, and replacing a, b, c, d by $a, b - c, c - d, d$. So we obtain the following four algebras:

$$\langle a, b, c, d \,|\, cb,\ da,\ db,\ dc,\ pa,\ pb,\ pc,\ pd - ca,\ \text{class } 2\rangle, \tag{6.29}$$

$$\langle a, b, c, d \mid cb,\ da,\ db,\ dc,\ pa - ba,\ pb,\ pc,\ pd - ca,\ \text{class } 2\rangle, \tag{6.30}$$

$$\langle a, b, c, d \mid cb,\ da,\ db,\ dc,\ pa,\ pb - ba,\ pc,\ pd - ca,\ \text{class } 2\rangle, \tag{6.31}$$

$$\langle a, b, c, d \mid cb,\ da,\ db,\ dc,\ pa,\ pb,\ pc - ba,\ pd - ca,\ \text{class } 2\rangle. \tag{6.32}$$

To see that these algebras are distinct, ørst note that $B = \langle b, c, d\rangle + L^2$ and $D = \langle d\rangle + L^2$ are characteristic subalgbras. In 6.29 pL has order p, but in 6.30 ˘ 6.32 it has order p^2. In 6.30 pB has order p, but in 6.31 and 6.32 it has order p^2. If we let $M = L/pD$ then $\langle c + pD, d + pD\rangle + M^2$ is the centre of M, and in 6.31 this centre has characteristic p, whereas in 6.32 it does not.

Case 2 Now consider the case when $cb = da = dc = 0,\ db = ba$. If $pL = \{0\}$ then we have

$$\langle a, b, c, d \mid cb,\ da,\ db - ba,\ dc,\ pa,\ pb,\ pc,\ pd,\ \text{class } 2\rangle. \tag{6.33}$$

Next, consider the case when L has this commutator structure, and pL has order p. It is straightforward to show that if a', b', c', d' is any other set of generators of L satisfying these commutator relations then we either have

$$\begin{aligned} a' &= \alpha a - \nu b + \beta c + (\alpha - \xi)d, \\ b' &= \gamma b + \delta d, \\ c' &= \lambda a + \mu c + \lambda d, \\ d' &= \nu b + \xi d \end{aligned} \tag{(*)}$$

modulo L^2, for some $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \xi$ with $\alpha\mu - \beta\lambda \neq 0$ and $\gamma\xi - \delta\nu \neq 0$, or we have

$$\begin{aligned} a' &= -\nu a + \alpha b - \xi c + \beta d, \\ b' &= \gamma a + \delta c + \gamma d, \\ c' &= \lambda b + \mu d, \\ d' &= \nu a + \xi c + \nu d \end{aligned} \tag{(**)}$$

modulo L^2, for some $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \xi$ with $\alpha\mu - \beta\lambda - \lambda\nu \neq 0$ and $\gamma\xi - \delta\nu \neq 0$. In the ørst of these

$$\begin{aligned} b'a' &= (\gamma\xi - \delta\nu)ba, \\ c'a' &= (\alpha\mu - \beta\lambda)ca, \end{aligned}$$

and in the second

$$\begin{aligned} b'a' &= (\gamma\xi - \delta\nu)ca, \\ c'a' &= (\alpha\mu - \beta\lambda - \lambda\nu)ba. \end{aligned}$$

So we may assume that $pL \leq Sp\langle ba\rangle$, or that $pL \leq Sp\langle ba + ca\rangle$.

Suppose that $pL \leq Sp\langle ba \rangle$. Then if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d, and if $pL \leq Sp\langle b'a' \rangle$, then a', b', c', d' satisfy (*) modulo L^2. Note that this implies that $Sp\langle b', d' \rangle = Sp\langle b, d \rangle$, so that we may assume that $pb = pd = 0$, or that $pd = 0$, $pb \neq 0$. If $pb = pd = 0$ then we can choose $\alpha, \beta, \lambda, \mu$ in (*) so that $pa \neq 0$, $pc = 0$. Scaling, we may assume that $pa = ba$, and we have

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa - ba,\ pb,\ pc,\ pd,\ \text{class } 2 \rangle. \tag{6.34}$$

And if $pd = 0$, $pb \neq 0$, then again we can choose $\alpha, \beta, \lambda, \mu$ in (*) so that $pc = 0$. Scaling, we may assume that $pa = 0$ or ba, and that $pb = ba$. So we have

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa,\ pb - ba,\ pc,\ pd,\ \text{class } 2 \rangle, \tag{6.35}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa - ba,\ pb - ba,\ pc,\ pd,\ \text{class } 2 \rangle. \tag{6.36}$$

Next, suppose that $pL \leq Sp\langle ba + ca \rangle$. Now, if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d, and if $pL \leq Sp\langle b'a' + c'a' \rangle$, then a', b', c', d' satisfy either (*) or (**) modulo L^2. As above, considering a', b', c', d' satisfying (*) we may assume that one of the following holds

$$\begin{aligned} pa &\neq 0,\ pb = pc = pd = 0, \\ pb &\neq 0,\ pa = pc = pd = 0, \\ pa &\neq 0,\ pb \neq 0,\ pc = pd = 0. \end{aligned}$$

However if we suppose $pb \neq 0$, $pa = pc = pd = 0$, then $a' = -a + b$, $b' = -c$, $c' = d$, $d' = a + d$ satisfy (**) and also satisy $pa' \neq 0$, $pb' = pc' = pd' = 0$. So, scaling we have

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa - ba - ca,\ pb,\ pc,\ pd,\ \text{class } 2 \rangle, \tag{6.37}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa - ba - ca,\ pb - ba - ca,\ pc,\ pd,\ \text{class } 2 \rangle. \tag{6.38}$$

Now let $cb = da = dc = 0$, $db = ba$ and let $pL = L^2$. First consider the case when pb, pd are linearly independant. Replacing a, b, c, d by suitable generators a', b', c', d' satisfying (*) we may assume that $pb = ba$, $pd = ca$. If we now consider possible generating sets satisfying (*), and in addition satisfying $pb' = b'a'$, $pd' = c'a'$, then we require $\delta = \nu = 0$, $\xi = 1$, $\alpha\mu - \beta\lambda = 1$. Now write

$$\begin{pmatrix} pa \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix},$$

with A a 2×2 matrix. If we replace a, b, c, d by a', b', c', d' where

$$\begin{aligned} a' &= \alpha a + \beta c + (\alpha - 1)d, \\ b' &= \gamma b, \\ c' &= \lambda a + \mu c + \lambda d, \\ d' &= d \end{aligned}$$

and $\alpha\mu - \beta\lambda = 1$, then

$$A \mapsto \begin{pmatrix} \alpha & \beta \\ \lambda & \mu \end{pmatrix} A \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha - 1 \\ 0 & \lambda \end{pmatrix}.$$

If the ørst column of A is non-zero, then we can pick $\alpha, \beta, \gamma, \lambda, \mu$ so that

$$A \mapsto \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}$$

for some x, y, and then taking $\alpha = \gamma$, $\mu = \alpha^{-1}$, $\lambda = 0$ we can transform this further to

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha - 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \alpha(x+1) + \beta y - 1 \\ 0 & \alpha^{-1} y \end{pmatrix}.$$

So we may assume that $x = 0$, $y = 1$, or that $y = 0$ and $x = 0$ or -1.

Now consider the case when the ørst column of A is zero, so that $A = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ for some x, y. Then

$$A \mapsto \begin{pmatrix} \alpha & \beta \\ \lambda & \mu \end{pmatrix} A \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha - 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & \alpha(x+1) + \beta y - 1 \\ 0 & \lambda(x+1) + \mu y \end{pmatrix}.$$

If $x \neq -1$ then we can take $\beta = 0$, $\mu = \alpha^{-1}$, and choose α, λ so that A transforms to the zero matrix. And if $x = -1$ and $y \neq 0$ then we can take $\mu = 0$ and transform A to the zero matrix. So we can take A to be one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we obtain

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa - ba,\ pb - ba,\ pc - ca,\ pd - ca,\ \text{class } 2\rangle, \tag{6.39}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa - ba,\ pb - ba,\ pc,\ pd - ca,\ \text{class } 2\rangle, \tag{6.39}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa - ba + ca,\ pb - ba,\ pc,\ pd - ca,\ \text{class } 2\rangle, \tag{6.40}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa + ca,\ pb - ba,\ pc,\ pd - ca,\ \text{class } 2\rangle, \tag{6.41}$$

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ba,\ dc,\ pa,\ pb - ba,\ pc,\ pd - ca,\ \text{class } 2\rangle. \tag{6.42}$$

We have now covered the case when pb, pd are linearly independant. If $pb = pd = 0$ then pa, pc must be linearly independant (since we are assuming that pL has order p^2). But now if we let a', b', c', d' satisfy (**) then pb', pd' are linearly independant, and we are back to the previous case. So we may assume that pb and pd span a space of dimension one, and by a change of generating set of the form (*) we may assume that $pb \neq 0$, $pd = 0$. If pa, pc are linearly independant and a', b', c', d' satisfy (**) then pb', pd' are linearly independant, and we are back to the previous case. So we may assume that pa and pc span a space of dimension one, and by a change of generating set of the form (*) with $\nu = 0$ we may assume that $pa \neq 0$, $pc = 0$. Summarizing, we have $pc = pd = 0$, and using the fact that we are assuming that $pL = L^2$ we have pa and pd linearly independant. Let

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} ba \\ ca \end{pmatrix},$$

where $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ has rank 2. If we consider possible generating sets a', b', c', d' satisfying (*), with $pc' = pd' = 0$ then we have

$$\begin{aligned} a' &= \alpha a + \beta c + (\alpha - \xi) d, \\ b' &= \gamma b + \delta d, \\ c' &= \mu c, \\ d' &= \xi d \end{aligned}$$

modulo L^2, with α, γ, μ, ξ all non-zero. It is straightforward to check that such a change of generating set transforms

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} \gamma^{-1}\xi^{-1} & 0 \\ 0 & \alpha^{-1}\mu^{-1} \end{pmatrix}.$$

And if we consider possible generating sets a', b', c', d' satisfying (**), with $pc' = pd' = 0$ then

$$\begin{aligned} a' &= \alpha b - \xi c + \beta d, \\ b' &= \gamma a + \delta c + \gamma d, \\ c' &= \mu d, \\ d' &= \xi c \end{aligned}$$

with α, γ, μ, ξ all non-zero. This change of generating set transforms

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} u & t \\ s & r \end{pmatrix} \begin{pmatrix} \gamma^{-1}\xi^{-1} & 0 \\ 0 & \alpha^{-1}\mu^{-1} \end{pmatrix}.$$

If none of r, s, t, u are zero then we can use either of these types of transformations to transform

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & u \end{pmatrix},$$

where $u \neq 0$. Since the rank of the matrix is two, we also have $u \neq 1$. If $r = 0$, but s, t, u are non-zero then we can transform

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with the ørst type of transformation, or to $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ with the second type. Similar considerations apply if any one of r, s, t, u is zero, with the other three being non-zero. If two of r, s, t, u are zero, then since the rank of $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ is two, we can transform $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So we obtain $p-2$ algebras

$$\langle a, b, c, d \mid cb,\ da,\ db - ba,\ dc,\ pa - ba - ca,\ pb - ba - uca,\ pc,\ pd,\ \text{class } 2\rangle\ (u \neq 0, 1), \tag{6.43}$$

and four other algebras

$$\langle a, b, c, d \mid cb,\ da,\ db - ba,\ dc,\ pa - ba - ca,\ pb - ba,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.44}$$

$$\langle a, b, c, d \mid cb,\ da,\ db - ba,\ dc,\ pa - ba - ca,\ pb - ca,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.45}$$

$$\langle a, b, c, d \mid cb,\ da,\ db - ba,\ dc,\ pa - ba,\ pb - ca,\ pc,\ pd,\ \text{class } 2\rangle, \tag{6.46}$$

$$\langle a, b, c, d \mid cb,\ da,\ db - ba,\ dc,\ pa - ca,\ pb - ba,\ pc,\ pd,\ \text{class } 2\rangle. \tag{6.47}$$

Case 3 Now we consider the case when $cb = da = dc = 0$, $db = ca$. It is straightforward to see that if a', b', c', d' generate L, and satisfy the same relations as a, b, c, d then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \varepsilon a + \zeta b + \eta c + \theta d, \\ c' &= \lambda\zeta c - \lambda\varepsilon d, \\ d' &= -\lambda\beta c + \lambda\alpha d \end{aligned}$$

modulo L^2 for some $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \lambda$ with $\alpha\zeta - \beta\varepsilon \neq 0$, $\lambda \neq 0$. If a', b', c', d' are as above then

$$\begin{aligned} b'a' &= (\alpha\zeta - \beta\varepsilon)ba + (\alpha\eta + \beta\theta - \gamma\varepsilon - \delta\zeta)ca, \\ c'a' &= \lambda(\alpha\zeta - \beta\varepsilon)ca. \end{aligned}$$

If $pL = 0$ then we have

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ pa,\ pb,\ pc,\ pd,\ \text{class } 2\rangle. \tag{6.48}$$

Next, consider the case when pL has order p. Clearly we may suppose that pL is spanned either by ba, or by ca.

Suppose for the moment that pL is spanned by ba. If $pc = pd = 0$ then we may choose $\alpha, \beta, \varepsilon, \zeta$ so that $pa' \neq 0$, $pb' = 0$. So we may assume that $pb = pc = pd = 0$, and by scaling we may assume that $pa = ba$. This gives

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ pa - ba,\ pb,\ pc,\ pd,\ \text{class } 2\rangle. \tag{6.49}$$

On the other hand, if pb and pc are not both zero, then we can choose $\alpha, \beta, \varepsilon, \zeta$ so that $pc' \neq 0$, $pd' = 0$. So we assume that $pc = ba$, $pd = 0$. Now, write $pa = -\gamma ba$, $pb = -\eta ba$ and let

$$\begin{aligned} a' &= a + \gamma c + \eta d, \\ b' &= b + \eta c, \\ c' &= c, \\ d' &= d. \end{aligned}$$

Then a', b', c', d' satisfy the same commutator relations as a, b, c, d and $pa' = pb' = pd' = 0$, $pc' = b'a'$. So we have

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ca,\ dc,\ pa,\ pb,\ pc - ba,\ pd, \text{ class } 2\rangle. \tag{6.50}$$

Next, consider the case when pL is spanned by ca. If $pc = pd = 0$ then just as above we may suppose that $pb = 0$, $pa = ca$, giving

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ca,\ dc,\ pa - ca,\ pb,\ pc,\ pd, \text{ class } 2\rangle. \tag{6.51}$$

And if pc, pd are not both zero then we may suppose that $pc = ca$, $pd = 0$, and just as above we may suppose that $pa = pb = 0$. So we have

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ca,\ dc,\ pa,\ pb,\ pc - ca,\ pd, \text{ class } 2\rangle. \tag{6.52}$$

Now consider the case when $pL = L^2$. If $pc = pd = 0$ then we can assume that $pa = ba$, $pb = ca$, giving

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ca,\ dc,\ pa - ba,\ pb - ca,\ pc,\ pd, \text{ class } 2\rangle. \tag{6.53}$$

If pc, pd span a space of dimension one then we can assume that $pd = 0$ and that $pc = ba$ or $pc = ca$. If $pc = ba$, then (as above) we let

$$\begin{aligned} a' &= a + \gamma c + \eta d, \\ b' &= b + \eta c, \\ c' &= c, \\ d' &= d. \end{aligned}$$

where γ, η are chosen so that pa' and pb' are linear multiples of ca. If $pa \neq 0$ then we can replace b by $b + \varepsilon a$ (and at the same time replace c by $c - \varepsilon d$), where ε is chosen so that $p(b + \varepsilon a) = 0$. Then, by scaling, we may assume that $pa = ca$, $pb = 0$, or we may assume that $pa = 0$, $pb = ca$, or that $pa = 0$, $pb = \omega ca$. Similarly, if $pc = ca$ then we can asume that $pa = ba$, $pb = 0$, or that $pa = 0$, $pb = kba$ for some $k \neq 0$. So we have

$$\langle a, b, c, d \,|\, cb,\ da,\ db - ca,\ dc,\ pa - ca,\ pb,\ pc - ba,\ pd, \text{ class } 2\rangle, \tag{6.54}$$

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ pa,\ pb - ca,\ pc - ba,\ pd,\ \text{class } 2\rangle, \tag{6.55}$$

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ pa,\ pb - \omega ca,\ pc - ba,\ pd,\ \text{class } 2\rangle, \tag{6.56}$$

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ pa - ba,\ pb,\ pc - ca,\ pd,\ \text{class } 2\rangle, \tag{6.57}$$

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ pa,\ pb - kba,\ pc - ca,\ pd,\ \text{class } 2\rangle\ (k = 1, 2, \ldots, p - 1). \tag{6.58}$$

Finally consider the case when pc, pd span a space of dimension 2. Then replacing a by $a + \gamma c + \delta d$ and replacing b by $b + \eta c + \theta d$ for suitable $\gamma, \delta, \eta, \theta$ we may suppose that $pa = pb = 0$. Then we can find new generators a', b', c', d' with

$$\begin{aligned}
a' &= \alpha a + \beta b,\\
b' &= \varepsilon a + \zeta b,\\
c' &= \lambda \zeta c - \lambda \varepsilon d,\\
d' &= -\lambda \beta c + \lambda \alpha d
\end{aligned}$$

such that $pc' = b'a'$, $pd' = c'a'$, giving

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc,\ pa,\ pb,\ pc - ba,\ pd - ca,\ \text{class } 2\rangle. \tag{6.59}$$

Case 4 Finally consider the case when $cb = da = 0$, $db = ca$, $dc = \omega ba$. We consider possible generating sets a', b', c', d' for L, where a', b', c', d' satisfy the same commutator relations as a, b, c, d. It is straightforward to show that if $a' = a$, $b' = b$, then we either have $c' = c$ and $d' = d$ modulo L^2 or we have $c' = -c$ and $d' = -d$ modulo L^2. It is also straightforward to show that if $a' = a$ then d' must centralize a, so that $d' = \alpha a + \beta d$ modulo L^2 for some α, β with $\beta \neq 0$. Finally, it is straightforward to show that if we let

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c + \delta d,\\
b' &= \lambda a + \mu b + \nu c + \xi d,\\
c' &= -\omega \xi a + \omega \nu b + \mu c - \lambda d,\\
d' &= \omega \delta a - \omega \gamma b - \beta c + \alpha d
\end{aligned}$$

then a', b', c', d' satisfy the same commutator relations as a, b, c, d. It follows that we can take a' to be any element outside L^2, and (having fixed a') we can take b' to be any element outside the centralizer of a'. This fixes c', d' up to change of sign.

If $pa = pb = pc = pd = 0$ then we have

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc - \omega ba,\ pa,\ pb,\ pc,\ pd,\ \text{class } 2\rangle. \tag{6.60}$$

If pa, pb, pc, pd span a space of dimension 1 then we can choose a so that $pa \neq 0$. Then, replacing a, d by suitable elements in the span of a, d we may suppose that $pa \neq 0$, $pd = 0$. Then, taking $\beta = \gamma = \delta = \nu = 0$, $\alpha = \mu = 1$ in the above expressions for a', b', c', d', and with suitable choice of λ, ξ, we can assume $pb = pc = pd = 0$. Finally, replacing b by $\mu b + \nu c$ for suitable μ, ν (at the same time as replcing c by $\omega\nu b + \mu c$), we may suppose that $pa = ba$. This gives

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc - \omega ba,\ pa - ba,\ pb,\ pc,\ pd,\ \text{class } 2\rangle. \qquad (6.60\text{B})$$

Now suppose that pa, pb, pc, pd span a space of dimension 2. Then there is a two dimensional subspace $U \leq Sp\langle a, b, c, d\rangle$ with $pu = 0$ for all $u \in U$. If the elements of U commute with each other, then we can suppose that U is spanned by a, d, so that $pa = pd = 0$ and pb, pc span a space of dimension 2. And if the elements of U do not all commute with each other then we may suppose that U is spanned by a, b, so that $pa = pb = 0$ and pc, pd are linearly independant.

First, consider the case when $pa = pb = 0$, pc, pd are linearly independant. Then if we set

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \lambda a + \mu b, \\ c' &= \mu c - \lambda d, \\ d' &= -\beta c + \alpha d \end{aligned}$$

for some $\alpha, \beta, \lambda, \mu$ with $\alpha\mu - \beta\lambda \neq 0$, then a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d. Furthermore $pa' = pb' = 0$ and pc', pd' are linearly independant. It is straightforward to check that

$$\begin{aligned} b'a' &= (\alpha\mu - \beta\lambda)ba, \\ c'a' &= (\alpha\mu - \beta\lambda)ca. \end{aligned}$$

So we can choose $\alpha, \beta, \lambda, \mu$ so that $pc' = xb'a'$, $pd' = yc'a'$ for some x, y. So we may suppose that $pa = pb = 0$, $pc = xba$, $pd = yca$. Then letting $a' = \alpha a$, $b' = \mu b$, $c' = \mu c$, $d' = \alpha d$, we have

$$\begin{aligned} pc' &= \mu pc = \mu x ba = x\alpha^{-1} b'a', \\ pd' &= \alpha pd = \alpha y ca = y\mu^{-1} c'a'. \end{aligned}$$

So we may suppose that $pa = pb = 0$, $pc = ba$, $pd = ca$, giving

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc - \omega ba,\ pa,\ pb,\ pc - ba,\ pd - ca,\ \text{class } 2\rangle. \qquad (6.61)$$

Finally, consider the case when $pa = pd = 0$, pb, pc are linearly independant. We let

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some non-singular 2×2 matrix A. If a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d and in addition satisfy $pa' = pd' = 0$ then

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \lambda a + \mu b + \nu c + \xi d, \\ c' &= \pm(-\omega \xi a + \omega \nu b + \mu c - \lambda d), \\ d' &= \pm(\omega \delta a + \alpha d) \end{aligned}$$

modulo L^2. It is straightforward to check that this implies that

$$\begin{pmatrix} b'a' \\ c'a' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} \begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix} \begin{pmatrix} ba \\ ca \end{pmatrix}$$

and that

$$\begin{pmatrix} pb' \\ pc' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix}.$$

So, if we replace a, b, c, d by a', b', c', d' then the matrix A transforms to

$$\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} A \begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix}^{-1} \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

For the moment consider the equivalence relation $\sim$ on the set of non-singular 2×2 matrices given by the rule $A \sim B$ if

$$B = \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} A \begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix}^{-1} \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}^{-1}$$

for some non-singular matrices $\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}$ and $\begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix}$. Using the fact that non-singular matrices $\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}$ form a group under matrix multiplication, we see that $A \sim B$ if

$$B = \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} A \begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix}^{-1}$$

for some non-singular matrices $\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}$ and $\begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix}$. Now note that if A is non-singular, then we can find μ, ν so that

$$\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} A = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$$

for some x, y. So every equivalence class contains a matrix of the form $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$. Next, we investigate the conditions under which $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$ can be equivalent to

$\begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix}$. This implies that

$$\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & t \end{pmatrix} \begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix}$$

for some μ, ν, α, δ. But then $\alpha = \mu + \nu x$, and $\delta = -\nu y$. So

$$\begin{pmatrix} \alpha & -\delta \\ -\omega\delta & \alpha \end{pmatrix} = \begin{pmatrix} \mu + \nu x & \nu y \\ \omega\nu y & \mu + \nu x \end{pmatrix}.$$

(Note that this matrix is non-singular whenever at least one of μ, ν is non-zero.) So the group of non-singular matrices $\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}$ acts as a group of order $p^2 - 1$ on the set of non-singular matrices of the form $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$, sending

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \to \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} \mu + \nu x & \nu y \\ \omega\nu y & \mu + \nu x \end{pmatrix}^{-1}$$

The stabilizer of the matrix $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$ is the set of matrices $\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}$ such that

$$\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} \mu + \nu x & \nu y \\ \omega\nu y & \mu + \nu x \end{pmatrix}.$$

So $\begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}$ stabilizes $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$ if

$$\begin{aligned} \omega\nu + \mu x &= \mu x + \nu x^2 + \omega\nu y^2, \\ \mu y &= \mu y + 2x\nu y. \end{aligned}$$

Note that $y \neq 0$ since $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$ is non-singular. So if $x \neq 0$ or $y \neq \pm 1$ the stabilizer has order $p - 1$. But if $x = 0$ and $y = \pm 1$ then the stabilizer is the whole group. So there are 2 orbits of size 1, and the remaining orbits have size $p + 1$: this implies that there are p orbits in all.

Now we have to consider the fact that there is a change of generating set for L which transforms

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -x & y \end{pmatrix}.$$

However, it turns out that

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ -x & y \end{pmatrix},$$

since

$$\begin{pmatrix} -\omega - x^2 + y^2\omega & 2x \\ 2x\omega & -\omega - x^2 + y^2\omega \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} -\omega + x^2 + y^2\omega & 2xy \\ 2xy\omega & -\omega + x^2 + y^2\omega \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -x & y \end{pmatrix}$$

So the p equivalence classes of non-singular matrices $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$ under the equivalence relation

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \sim \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} \mu + \nu x & \nu y \\ \omega\nu y & \mu + \nu x \end{pmatrix}^{-1}$$

correspond to non-isomorphic algebras. Thus we have the following algebras (with x, y to correspond to representives of these equivalence classes)

$$\langle a, b, c, d \mid cb,\ da,\ db - ca,\ dc - \omega ba,\ pa,\ pb - ba,\ pc - xba - yca,\ pd,\ \text{class } 2\rangle. \quad (6.62)$$

3.2 Descendants of 5.2

Let L be an immediate descendant of 5.2. Then L is generated by a, b, c, d, L_2 is spanned by pa modulo L_3, and L_3 is spanned by p^2a. We have $pb, pc, pd \in L_3$, and all commutators lie in L_3. Replacing b by $b - \alpha pa$ for suitable α we may assume that $pb = 0$, and we may similarly assume that $pc = pd = 0$. Note that $B = \langle b, c, d\rangle + L_3$ is a characteristic subalgebra, and that B is abelian if and only if $\langle b, c, d\rangle$ is abelian.

3.2.1 L abelian

If L is abelian then we have

$$\langle a, b, c, d \mid ba,\ ca,\ da,\ cb,\ db,\ dc,\ pb,\ pc,\ pd,\ \text{class } 3\rangle. \quad (6.63)$$

3.2.2 $\langle b, c, d\rangle$ abelian

If $\langle b, c, d\rangle$ is abelian, but L is not abelian then we may assume that $ba = p^2a$, and that $ca = da = 0$, so we have

$$\langle a, b, c, d \mid ba - p^2a,\ ca,\ da,\ cb,\ db,\ dc,\ pb,\ pc,\ pd,\ \text{class } 3\rangle. \quad (6.64)$$

3.2.3 $\langle b, c, d\rangle$ non-abelian

If $\langle b, c, d\rangle$ is not abelian, then we may assume that $cb = p^2a$, $db = dc = 0$. Replacing a by $a - \beta b - \gamma c$ for suitable β, γ we may suppose that $ba = ca = 0$. Scaling d we may suppose that $da = 0$ or p^2a. So we have

$$\langle a, b, c, d \mid ba,\ ca,\ da,\ cb - p^2a,\ db,\ dc,\ pb,\ pc,\ pd,\ \text{class } 3\rangle, \quad (6.65)$$

$$\langle a, b, c, d \mid ba,\ ca,\ da - p^2a,\ cb - p^2a,\ db,\ dc,\ pb,\ pc,\ pd,\ \text{class } 3\rangle. \quad (6.66)$$

These two algebras are distinct, since in 6.65 the centre has order p^4, but in 6.66 it has order p^2.

3.3 Descendants of 5.3

Let L be an immediate descendant of 5.3. Then L is generated by a, b, c, d, L_2 is spanned by ba modulo L_3, and L_3 is spanned by baa, bab. We have $pa, pb, pc, pd \in L_3$, and all commutators other than ba lie in L_3. The commutator structure of L must correspond to one of the algebras 6.8 ˘ 6.10 from the list of nilpotent Lie algebras over $\mathbb{Z}_p$ of dimension 6. So we may assume that one of the following sets of commutator relations holds.

$$\begin{aligned} bab &= ca = da = cb = db = dc = 0, \\ bab &= cb = da = db = dc = 0,\ cb = baa, \\ bab &= ca = da = cb = db = 0,\ dc = baa. \end{aligned}$$

Note that algebras satisfying these three sets of commutator relations have centres of order p^3, p^2 and p respectively.

3.3.1 Case 1

First suppose that $bab = ca = da = cb = db = dc = 0$. Then c and d are central. If $pc = pd = 0$ then

$$L = M \oplus \langle c \mid pc = 0 \rangle \oplus \langle d \mid pd = 0 \rangle$$

where M is an immediate descendant of 3.2 of order p^4. So we have

$$\langle a, b, c, d \mid bab, ca, cb, da, db, dc,\ pa,\ pb, pc, pd,\ \text{class } 3 \rangle, \tag{6.67}$$

$$\langle a, b, c, d \mid bab, ca, cb, da, db, dc,\ pa - baa,\ pb, pc, pd,\ \text{class } 3 \rangle, \tag{6.68}$$

$$\langle a, b, c, d \mid bab, ca, cb, da, db, dc,\ pa,\ pb - baa, pc, pd,\ \text{class } 3 \rangle, \tag{6.69}$$

$$\langle a, b, c, d \mid bab, ca, cb, da, db, dc,\ pa,\ pb - \omega baa, pc, pd,\ \text{class } 3 \rangle. \tag{6.70}$$

On the other hand, if pc, pd are not both zero then we may assume that $pc = baa$, $pd = 0$. Then subtracting suitable multiples of c from a, b we may suppose that $pa = pb = 0$. So we have

$$\langle a, b, c, d \mid bab, ca, cb, da, db, dc,\ pa,\ pb, pc - baa, pd,\ \text{class } 3 \rangle. \tag{6.71}$$

The algebras 6.67 ˘ 6.70 are distinct, since they are of the form $M \oplus \langle c \mid pc = 0 \rangle \oplus \langle d \mid pd = 0 \rangle$ with distinct M. And 6.71 is diœerent from 6.67 ˘ 6.70 since in 6.67 ˘ 6.70 the centre $\langle c, d, baa \rangle$ has characteristic p, and in 6.71 $pc \neq 0$.

3.3.2 Case 2

Now suppose that $bab = cb = da = db = dc = 0$, $cb = baa$. Here d is again central and $\langle a, b, c\rangle$ is isomorphic to one of 5.32 ˘ 5.36. We may suppose that $pd = 0$ or baa, but if $pd = baa$ then by subtracting suitable scalar multiples of d from a, b, c we may suppose that $pa = pb = pc = 0$ so that $\langle a, b, c\rangle$ is isomorphic to 5.32. So we have

$$\langle a, b, c, d \,|\, ca,\ cb - baa,\ bab,\ da,\ db,\ dc,\ pa,\ pb,\ pc,\ pd,\ \text{class } 3\rangle, \tag{6.72}$$

$$\langle a, b, c, d \,|\, ca,\ cb - baa,\ bab,\ da,\ db,\ dc,\ pa - baa,\ pb,\ pc,\ pd,\ \text{class } 3\rangle, \tag{6.73}$$

$$\langle a, b, c, d \,|\, ca,\ cb - baa,\ bab,\ da,\ db,\ dc,\ pa,\ pb - baa,\ pc,\ pd,\ \text{class } 3\rangle, \tag{6.74}$$

$$\langle a, b, c, d \,|\, ca,\ cb - baa,\ bab,\ da,\ db,\ dc,\ pa,\ pb - \omega baa,\ pc,\ pd,\ \text{class } 3\rangle, \tag{6.75}$$

$$\langle a, b, c, d \,|\, ca,\ cb - baa,\ bab,\ da,\ db,\ dc,\ pa,\ pb,\ pc - baa,\ pd,\ \text{class } 3\rangle, \tag{6.76}$$

$$\langle a, b, c, d \,|\, ca,\ cb - baa,\ bab,\ da,\ db,\ dc,\ pa,\ pb,\ pc,\ pd - baa,\ \text{class } 3\rangle. \tag{6.77}$$

The algebras 6.72 ˘ 6.76 are distinct since they have the form $M \oplus \langle d \,|\, pd = 0\rangle$ with distinct M. And 6.77 is distinct from 6.72 ˘ 6.76 since in 6.72 ˘ 6.76 the centre $\langle d, baa\rangle$ has characteristic p, and in 6.77 $pd \neq 0$.

3.3.3 Case 3

Finally suppose that $bab = ca = da = cb = db = 0$, $dc = baa$. Clearly the subalgebra generated by a, b is isomorphic to one of 4.9 ˘ 4.12, so if $pc = pd = 0$ then we have

$$\langle a, b, c, d \,|\, bab, ca, cb, da, db, dc - baa,\ pa,\ pb, pc, pd,\ \text{class } 3\rangle, \tag{6.78}$$

$$\langle a, b, c, d \,|\, bab, ca, cb, da, db, dc - baa,\ pa - baa,\ pb, pc, pd,\ \text{class } 3\rangle, \tag{6.79}$$

$$\langle a, b, c, d \,|\, bab, ca, cb, da, db, dc - baa,\ pa,\ pb - baa, pc, pd,\ \text{class } 3\rangle, \tag{6.80}$$

$$\langle a, b, c, d \,|\, bab, ca, cb, da, db, dc - baa,\ pa,\ pb - \omega baa, pc, pd,\ \text{class } 3\rangle. \tag{6.81}$$

If pc and pd are not both zero then we may assume that $pc = baa$, $pd = 0$. Replacing a by $a - \gamma c$, d by $d + \gamma ba$ for suitable γ we may suppose that $pa = 0$. So we have

$$\langle a, b, c, d \,|\, bab, ca, cb, da, db, dc - baa,\ pa,\ pb, pc - baa, pd,\ \text{class } 3\rangle, \tag{6.82}$$

$$\langle a, b, c, d \,|\, bab, ca, cb, da, db, dc - baa,\ pa,\ pb - baa, pc - baa, pd,\ \text{class } 3\rangle, \tag{6.83}$$

$$\langle a, b, c, d \,|\, bab, ca, cb, da, db, dc - baa,\ pa,\ pb - \omega baa, pc - baa, pd,\ \text{class } 3\rangle. \tag{6.84}$$

To show that these seven algebras are distinct we note that $B = \langle b, c, d\rangle + L^2$ and $C = \langle c, d\rangle + L^2$ are characteristic subalgbras. The algebra L has characteristic p in 6.78, but not in 6.79 ˘ 6.84. The subalgebra B has characteristic p in 6.79, but not in 6.80 ˘ 6.84. And C has characteristic p in 6.80 and 6.81, but not in 6.82 ˘ 6.84.

To see that 6.80 and 6.81 are distinct from each other we note that if $a' = \alpha a$ modulo B, and if $b' = \beta b$ modulo C, then in 6.80 we have

$$pb' = \beta pb = \alpha^{-2} b'a'a',$$

and in 6.81 we have

$$pb' = \beta pb = \alpha^{-2} \omega b'a'a'.$$

The same analysis shows that 6.82 ˘ 6.84 are distinct.

4 Three generators

If L is a three generator nilpotent Lie ring of order p^6 then L must be an immediate descendant of 3.1, or one of 4.2 ˘ 4.5, or one of 5.8 ˘5.36.

4.1 Descendants of 3.1

We subdivide the descendants of 3.1 according to the oder of L^2.

4.1.1 L is abelian

If L is abelian then L is

$$\langle a, b, c \mid ba, ca, cb, \text{ class } 2 \rangle. \tag{6.85}$$

4.1.2 L^2 has order p

If L^2 has order p then we may suppose that L^2 is spanned by ba, and that $ca = cb = 0$. If $pc = 0$ then we have

$$\langle a, b, c \mid ca, cb, pc, \text{ class } 2 \rangle, \tag{6.86}$$

and if $pc \neq 0$ but pc is a scalar multiple of ba then we may suppose that $pc = ba$, and we have

$$\langle a, b, c \mid ca, cb, pc - ba, \text{ class } 2 \rangle. \tag{6.87}$$

So suppose that $pc \notin \langle ba \rangle$. Clearly, pa, pb, pc span a two dimensional space modulo $\langle ba \rangle$, and so we may assume that pb, pc span a two dimensional space modulo $\langle ba \rangle$. Replacing a by $a - \beta b - \gamma c$ for suitable β, γ we may suppose that $pa \in \langle ba \rangle$. So we may assume that $pa = 0$ or ba. So we have

$$\langle a, b, c \mid ca, cb, pa, \text{ class } 2 \rangle, \tag{6.88}$$

$$\langle a, b, c \mid ca, cb, pa - ba, \text{ class } 2 \rangle. \tag{6.89}$$

To see that these four algebras are distinct we note that pL has order p^2 in 6.86 and 6.88, and order p^3 in 6.87 and 6.89. If we let $C = \langle c \rangle + L_2$, then C is the centre of L, and $pC \leq L^2$ in 6.86 and 6.87, but $pC \nleq L^2$ in 6.88 and 6.89.

4.1.3 L^2 has order p^2

If L^2 has order p^2 then we may suppose that L^2 is spanned by ba, ca, and that $cb = 0$. Note that $B = \langle b, c\rangle + L_2$ is a characteristic subalgebra of L. We consider separately the cases when pL has order p, p^2, or p^3.

pL has order p If $pB = \{0\}$ then we have

$$\langle a, b, c \,|\, cb,\ pb,\ pc,\ \text{class } 2\rangle. \tag{6.90}$$

On the other hand, if $pB \neq \{0\}$ then we can assume that $pa = pc = 0$, and we have

$$\langle a, b, c \,|\, cb,\ pa,\ pc,\ \text{class } 2\rangle. \tag{6.91}$$

pL has order p^2 If $pB \leq L^2$ then pB must have order p, and we may suppose that $pb = \beta ba + \gamma ca$ for some β, γ (not both zero), $pc = 0$. If $\beta \neq 0$ then we can replace b by $\beta b + \gamma c$, and after scaling a we can assume that $pb = ba$. And if $\beta = 0$ then after scaling a we can assume that $pb = ca$. So we have

$$\langle a, b, c \,|\, cb,\ pb - ba,\ pc,\ \text{class } 2\rangle, \tag{6.92}$$

$$\langle a, b, c \,|\, cb,\ pb - ca,\ pc,\ \text{class } 2\rangle. \tag{6.93}$$

To see that these two algebras are distinct, consider the subalgebra $C = \langle c\rangle + L_2$. This is the unique subalgebra of characteristic p and order p^4, and in 6.92 $pB \nleq CL$, whereas in 6.93 $pB \leq CL$.

So suppose that $pB \nleq L^2$. Then we may suppose that $pb \notin L^2$, and that $pc \in L^2$. Replacing a by $a - \lambda b$ for suitable λ we can assume that $pa \in L^2$. So pa, pc span a subspace of L^2 of dimension 1. If $pc \neq 0$ then we can replace a by $a - \mu c$ for suitable μ so that $pa = 0$. So we can assume that $pa = 0$ or that $pc = 0$. Suppose for the moment that $pa = 0$ and that $pc = \beta ba + \gamma ca$. If $\beta \neq 0$ then we can replace b by $\beta b + \gamma c$, so that $pc = ba$. And if $\beta = 0$ then after scaling a we can assume that $pc = ca$. So we have

$$\langle a, b, c \,|\, cb,\ pa,\ pc - ba,\ \text{class } 2\rangle, \tag{6.94}$$

$$\langle a, b, c \,|\, cb,\ pa,\ pc - ca,\ \text{class } 2\rangle. \tag{6.95}$$

Finally, suppose that $pc = 0$ and that $pa = \beta ba + \gamma ca$. If $\beta \neq 0$ then we can replace b by $\beta b + \gamma c$, so that $pa = ba$. And if $\beta = 0$ then after scaling c we can assume that $pa = ca$. So we have

$$\langle a, b, c \,|\, cb,\ pa - ba,\ pc,\ \text{class } 2\rangle, \tag{6.96}$$

$$\langle a, b, c \,|\, cb,\ pa - ca,\ pc,\ \text{class } 2\rangle. \tag{6.97}$$

In the four algebras 6.94 ˇ 6.97 the subalgebra $C = \langle c\rangle + L_2$ is a characteristic subalgebra since it is the unique subalgebra of B of order p^4 with $pC \leq L^2$. In

6.96 and 6.97 $pC = \{0\}$, whereas $pC \neq \{0\}$ in 6.94 and 6.95. Also $pC \leq CL$ in 6.95, but $pC \not\leq CL$ in 6.94. To distinguish 6.96 from 6.97 we let $D = \langle a, c\rangle + L_2$. This subalgebra is characteristic since it is the unique subalgebra D of order p^5 with $pD \leq L^2$. In 6.97 $pD \leq DL$ but in 6.96 $pD \not\leq DL$.

pL has order p^3 If $pB \leq L^2$ we may suppose that

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some non-singular matrix A. If we let

$$\begin{pmatrix} b' \\ c' \end{pmatrix} = P \begin{pmatrix} b \\ c \end{pmatrix}$$

for some non-singular matrix P then

$$\begin{pmatrix} pb' \\ pc' \end{pmatrix} = PAP^{-1} \begin{pmatrix} b'a \\ c'a \end{pmatrix},$$

and scaling a we have

$$\begin{pmatrix} pb' \\ pc' \end{pmatrix} = \lambda PAP^{-1} \begin{pmatrix} b'a \\ c'a \end{pmatrix} \ (\lambda \neq 0).$$

So by Theorem 6 we can choose P and λ so that A is one of the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} (\mu \neq 0),\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 0 & \mu \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - c$ is irreducible. Furthermore none of these matrices give isomorphic algebras, except that if $\mu \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$ gives an algebra which is isomorphic to that given by $\begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix}$. So we have $(p+1)/2$ algebras

$$\langle a, b, c \mid cb,\ pb - ba,\ pc - \mu ca,\ \text{class } 2\rangle \ (\mu \neq 0,\ \mu, \mu^{-1} \text{ give isomorphic algebras}), \tag{6.98}$$

$$\langle a, b, c \mid cb,\ pb - ba - ca,\ pc - ca,\ \text{class } 2\rangle, \tag{6.99}$$

$$\langle a, b, c \mid cb,\ pb - \omega ca,\ pc - ba,\ \text{class } 2\rangle, \tag{6.100}$$

and $(p-1)/2$ algebras

$$\langle a, b, c \mid cb,\ pb - \mu ca,\ pc - ba - ca,\ \text{class } 2\rangle \ (x^2 - x - \mu \text{ irreducible}). \tag{6.101}$$

If $pB \not\leq L^2$ then we may suppose that $pb \notin L^2$, $pc \in L^2$. And replacing a by $a - \lambda b$ for suitable λ we may supppose that $pa \in L^2$. Let $pc = \beta ba + \gamma ca$. If $\beta \neq 0$ then we can replace b by $\beta b + \gamma c$, so that $pc = ba$. And if $\beta = 0$ then we can scale a so that $pc = ca$. So we may assume that $pc = ba$ or ca. First consider the case when $pc = ba$. Then replacing a by $a - \mu c$ for suitable μ we may assume that $pa = \nu ca$. Scaling b and c we can take $\nu = 1$. So we have

$$\langle a, b, c \,|\, cb,\ pa - ca,\ pc - ba,\ \text{class } 2\rangle. \tag{6.102}$$

Similarly, if $pc = ca$ we have

$$\langle a, b, c \,|\, cb,\ pa - ba,\ pc - ca,\ \text{class } 2\rangle. \tag{6.103}$$

To see that 6.102 and 6.103 are distinct we note that the fact that B is characteristic and that $pb \in L^2$, $pc \in L^2$ implies that $C = \langle c\rangle + L_2$ is a characteristic subalgebra. In 6.102 $pC \not\leq CL$, but in 6.103 $pC \leq CL$.

4.1.4 L^2 has order p^3

$pL = \{0\}$ If $pL = 0$ then we have

$$\langle a, b, c \,|\, pa,\ pb,\ pc,\ \text{class } 2\rangle. \tag{6.104}$$

pL has order p If pL has order p then we may assume that $pb = pc = 0$. If $pa = \lambda cb$, then scaling a we can take $\lambda = 1$ and we have

$$\langle a, b, c \,|\, pa - cb,\ pb,\ pc,\ \text{class } 2\rangle. \tag{6.105}$$

If $pa = \lambda ba + \mu ca + \nu cb$ where either $\lambda \neq 0$ or $\mu \neq 0$ then we can assume that $pa = ba$, giving

$$\langle a, b, c \,|\, pa - ba,\ pb,\ pc,\ \text{class } 2\rangle. \tag{6.106}$$

Note that $B = \langle b, c\rangle + L^2$ is characteristic, and that $pL \leq B^2$ in 6.105, but $pL \not\leq B^2$ in 6.106.

pL has order p^2 If pL has order p^2 then we can assume that pa and pb are linearly independant, and that $pc = 0$. We let $C = \langle c\rangle + L^2$. Note that C is a characteristic subalgebra of L.

First suppose that $pa, pb \leq CL$. Then we can write

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} ca \\ cb \end{pmatrix}$$

for some non-singular matrix A. If we let

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = P \begin{pmatrix} a \\ b \end{pmatrix}$$

for some non-singular matrix P, and if we scale c then

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \lambda PAP^{-1} \begin{pmatrix} ca' \\ cb' \end{pmatrix} \ (\lambda \neq 0).$$

So by Theorem 6 we can take A to be one of the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} (\mu \neq 0),\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 0 & \mu \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - c$ is irreducible. Furthermore none of these matrices give isomorphic algebras, except that if $\mu \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$ gives an algebra which is isomorphic to that given by $\begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix}$. So we have $(p+1)/2$ algebras

$$\langle a, b, c \,|\, pa - ca,\ pb - \mu cb,\ pc,\ \text{class } 2\rangle\ (\mu \neq 0,\ \mu, \mu^{-1} \text{ give isomorphic algebras}), \tag{6.108}$$

$$\langle a, b, c \,|\, pa - ca - cb,\ pb - cb,\ pc,\ \text{class } 2\rangle, \tag{6.109}$$

$$\langle a, b, c \,|\, pa - \omega cb,\ pb - ca,\ pc,\ \text{class } 2\rangle, \tag{6.110}$$

and $(p-1)/2$ algebras

$$\langle a, b, c \,|\, pa - \mu cb,\ pb - ca - cb,\ pc,\ \text{class } 2\rangle\ (x^2 - x - \mu \text{ irreducible}). \tag{6.111}$$

Next, consider the case when pa, pb do not both lie in CL. Then we can suppose that $pa \notin CL$, $pb \in CL$. Let $pb = \alpha ca + \beta cb$. If $\alpha \neq 0$ then we can replace a by $\alpha a + \beta b$ so that $pb = ca$. And if $\alpha = 0$ we can scale c so that $pb = cb$. Assume for the moment that $pb = ca$. So we have

$$\begin{aligned} pa &= \lambda ba + \mu ca + \nu cb, \\ pb &= ca, \\ pc &= 0, \end{aligned}$$

for some λ, μ, ν with $\lambda \neq 0$. Now let

$$\begin{aligned} a' &= a - \nu\lambda^{-1}c, \\ b' &= \lambda b + \mu c, \\ c' &= \lambda c. \end{aligned}$$

Then

$$\begin{aligned} pa' &= b'a', \\ pb' &= c'a', \\ pc' &= 0. \end{aligned}$$

So we have

$$\langle a, b, c \,|\, pa - ba,\ pb - ca,\ pc,\ \text{class } 2\rangle. \tag{6.112}$$

Finally, consider the case when

$$\begin{aligned} pa &= \lambda ba + \mu ca + \nu cb, \\ pb &= cb, \\ pc &= 0, \end{aligned}$$

for some λ, μ, ν with $\lambda \neq 0$. Let

$$\begin{aligned} a' &= a - \nu\lambda^{-1}c, \\ b' &= \lambda b + \mu c, \\ c' &= c. \end{aligned}$$

Then

$$\begin{aligned} pa' &= b'a', \\ pb' &= c'b', \\ pc' &= 0. \end{aligned}$$

So we have

$$\langle a, b, c \,|\, pa - ba,\ pb - cb,\ pc,\ \text{class } 2\rangle. \tag{6.113}$$

pL **has order** p^3 Let $pa = \lambda ba + \mu ca + \nu cb$. If $\nu \neq 0$ we can scale a so that $\nu = 1$, and then replacing b by $b + \mu a$, and replacing c by $c - \lambda a$ we have $pa = cb$. On the other hand, if $\nu = 0$ we can choose b, c so that $pa = ba$. So we can always assume that $pa = cb$ or ba. We show that in fact we can always assume that $pa = ba$.

So suppose for the moment that $pa = cb$. Let

$$\begin{aligned} pb &= \alpha ba + \beta ca + \gamma cb, \\ pc &= \lambda ba + \mu ca + \nu cb. \end{aligned}$$

Note that if $\beta = 0$ then we can take $a' = b$, $b' = -\alpha a + \gamma c$ and then $pa' = b'a'$. Similarly, if $\lambda = 0$ we can take $a' = c$, $b' = -\mu a - \nu b$ and then $pa' = b'a'$. So we assume that $\beta \neq 0$, $\lambda \neq 0$, and we let

$$a' = a + rb + sc.$$

We show that we can always choose r, s so that $pa' \in La'$. This means that we can write $pa' = b'a'$ for some b', which achieves our aim.

So let a' be as deøned above. Then

$$\begin{aligned} pa' &= (r\alpha + s\lambda)ba + (r\beta + s\mu)ca + (1 + r\gamma + s\nu)cb, \\ ba' &= ba - scb, \\ ca' &= ca + rcb. \end{aligned}$$

So $pa' \in La'$ if

$$-s(r\alpha + s\lambda) + r(r\beta + s\mu) = 1 + r\gamma + s\nu.$$

We can rewrite this equation as

$$\beta(r + \rho)^2 - \lambda(s + \sigma)^2 = \tau$$

for some ρ, σ, τ, and then Lemma 1 implies that there exist r, s satisfying this equation.

So we may suppose that $pa = ba$. Let

$$\begin{aligned} pb &= \alpha ba + \beta ca + \gamma cb, \\ pc &= \lambda ba + \mu ca + \nu cb. \end{aligned}$$

and let

$$\begin{aligned} a' &= ka, \\ b' &= b - la, \\ c' &= ra + sb + tc. \end{aligned}$$

Then

$$\begin{aligned} pa' &= kba, \\ pb' &= (\alpha - l)ba + \beta ca + \gamma cb, \\ pc' &= (r + s\alpha + t\lambda)ba + (s\beta + t\mu)ca + (s\gamma + t\nu)cb, \\ b'a' &= kba, \\ c'a' &= ksba + ktca, \\ c'b' &= (-r - ls)ba - ltca + tcb. \end{aligned}$$

If $\gamma \neq 0$ then we let $t = \gamma$, $s = -\nu$, and we choose k so that $kt = s\beta + t\mu$, choose l so that $-lt = \beta$, and choose r so that $-r - ls = \alpha - l$. Then

$$\begin{aligned} pa' &= b'a', \\ pb' &= c'b', \\ pc' &= \delta b'a' + c'a' \end{aligned}$$

for some δ. This gives p algebras

$$\langle a, b, c \,|\, pa - ba,\ pb - cb,\ pc - kba - ca,\ \text{class } 2\rangle\ (k = 0, 1, \ldots, p-1). \qquad (6.114)$$

On the other hand, if $\gamma = 0$ then $\beta \neq 0$ since pa and pb must be linearly independant. For any given k, l we let $s = k^{-1}(\alpha - l)$, $t = k^{-1}\beta$ so that $pb' = c'a'$. So, replacing a, b, c by a', b', c' we have

$$\begin{aligned} pa &= ba, \\ pb &= ca, \\ pc &= \lambda ba + \mu ca + \nu cb \end{aligned}$$

for some λ, μ, ν. (Note the values of λ, μ, ν may have changed from those above.) We let

$$\begin{aligned} a' &= a, \\ b' &= b - la, \\ c' &= ra + sb + c, \end{aligned}$$

so that

$$\begin{aligned} pb' &= -lba + ca, \\ pc' &= (r + \lambda)ba + (s + \mu)ca + \nu cb, \\ b'a' &= ba, \\ c'a' &= sba + ca, \\ c'b' &= (-r - ls)ba - lca + cb. \end{aligned}$$

If we let $s = -l$ then

$$\begin{aligned} pa' &= b'a', \\ pb' &= c'a', \\ pc' &= (r + \lambda)ba + (-l + \mu)ca + \nu cb \\ &= (r + \lambda + r\nu - l^2\nu)ba + (-l + \mu + l\nu)ca + \nu c'b'. \end{aligned}$$

If $\nu \neq \pm 1$ we can choose r, l so that $pc' = \nu c'b'$. If $\nu = 1$ then we can choose r so that $pc' = \mu c'a' + c'b'$, and if $\nu = -1$ then we can choose $l = \mu/2$ so that $pc' = \lambda' b'a' - c'b'$ for some λ'. In other words, we can choose a, b, c so that

$$\begin{aligned} pa &= ba, \\ pb &= ca, \\ pc &= \nu cb \text{ or } \mu ca + cb \text{ or } \lambda ba - cb \end{aligned}$$

for some $\lambda, \mu, \nu \neq 0$. If the above relations hold, and we let $a' = ka$, $b' = b$, $c' = k^{-1}c$, then

$$\begin{aligned} pa' &= b'a', \\ pb' &= c'a', \\ pc' &= \nu c'b' \text{ or } \mu k^{-1}c'a' + c'b' \text{ or } \lambda k^{-2}b'a' - c'b'. \end{aligned}$$

So we can take $\mu = 1$ and $\lambda = 1$ or ω.

Consider the case when

$$\begin{aligned} pa &= ba, \\ pb &= ca, \\ pc &= ca + cb. \end{aligned}$$

Let $a' = c$, $b' = -a - b$, $c' = a$. Then

$$\begin{aligned} pa' &= ca + cb = b'a', \\ pb' &= -ba - ca = c'a' - c'b', \end{aligned}$$

and we are back to 6.114.

Next consider the case when

$$\begin{aligned} pa &= ba, \\ pb &= ca, \\ pc &= ba - cb. \end{aligned}$$

Let $a' = c - b$, $b' = a + c$, $c' = b$. Then

$$\begin{aligned} pa' &= ba - ca - cb = b'a', \\ pb' &= 2ba - cb = -c'a' + 2c'b', \end{aligned}$$

so we have 6.114 again.

Finally consider the case when

$$\begin{aligned} pa &= ba, \\ pb &= ca, \\ pc &= \nu cb, \end{aligned}$$

where $\nu \neq \pm 1$. We let

$$\begin{aligned} a' &= a + b + \frac{1}{\nu + 1}c, \\ b' &= b + c, \\ c' &= c, \end{aligned}$$

so that

$$\begin{aligned}
pa' &= ba + ca + \frac{\nu}{\nu+1}cb,\\
pb' &= ca + \nu cb,\\
b'a' &= ba + ca + \frac{\nu}{\nu+1}cb,\\
c'a' &= ca + cb,\\
c'b' &= cb.
\end{aligned}$$

This implies that $pa' = b'a'$, $pb' = c'a' + (\nu - 1)c'b'$, so again we are back to 6.114. This leaves the following possibilities:

$$\begin{aligned}
pa &= ba,\\
pb &= ca,\\
pc &= \pm cb \text{ or } \omega ba - cb
\end{aligned}$$

giving

$$\langle a, b, c \,|\, pa - ba,\ pb - ca,\ pc - cb,\ \text{class } 2\rangle, \tag{6.115}$$

$$\langle a, b, c \,|\, pa - ba,\ pb - ca,\ pc + cb,\ \text{class } 2\rangle, \tag{6.116}$$

$$\langle a, b, c \,|\, pa - ba,\ pb - ca,\ pc - \omega ba + cb,\ \text{class } 2\rangle. \tag{6.117}$$

We need to show that the $p+3$ algebras 6.114 ˘ 6.117 are distinct. First we show that none of the algebras 6.115 ˘ 6.117 have a set of generators a', b', c' with

$$pa' = b'a',\ pb' = c'b'. \tag{(*)}$$

This will show that 6.115 ˘ 6.117 cannot be isomorphic to any of the algbras 6.114. First note that if a', b', c' generate L and satisfy (*), then (*) still holds true if a' is replaced by any scalar multiple of itself.

Let L be the algebra 6.115, and suppose that a', b', c' satisfy (*). We aim to obtain a contradiction. First assume that $a' = a$. Then, to ensure that $pa' = b'a'$ we must have $b' = b + \alpha a$ for some α, so that $pb' = \alpha ba + ca$. Let $c' = \lambda a + \mu b + \nu c$ (where necessarily $\nu \neq 0$). Then

$$c'b' = (\alpha\mu - \lambda)ba + \alpha\nu ca + \nu cb$$

and since $\nu \neq 0$, $p'b' \neq c'b'$. So we cannot have such a generating set with $a' = a$, or with a' any scalar multiple of a.

Next consider the case when $a' = a + \beta b + \gamma c$. To ensure that $pa' \in a'L$ we need $\gamma = \beta^2/2$. Now let

$$\begin{aligned}
a' &= a + \beta b + \frac{\beta^2}{2}c,\\
b' &= b + \beta c,\\
c' &= c.
\end{aligned}$$

Then

$$
\begin{aligned}
pa' &= ba + \beta ca + \frac{\beta^2}{2} cb. \\
pb' &= ca + \beta cb, \\
pc' &= cb, \\
b'a' &= ba + \beta ca + \frac{\beta^2}{2} cb, \\
c'a' &= ca + \beta cb, \\
c'b' &= cb.
\end{aligned}
$$

So a', b', c' satisfy the same relations as a, b, c, and the argument above shows that there are no generators a', b'', c'' such that $pa' = b''a'$, $pb' = c''b''$. So there are no generators a', b', c' satisfying (*), where a' is a scalar multiple of $a + \beta b + \gamma c$.

It is easy to see that if $a' = b + \gamma c$ then $pa' \notin a'L$, so there are no generators a', b', c' satisfying (*) with a' equal to a scalar multiple of $b + \gamma c$.

Finally, let $a' = c$, $b' = -b$, $c' = a$. Then

$$
\begin{aligned}
pa' &= cb = b'a', \\
pb' &= -ca = c'a', \\
pc' &= ba = c'b'.
\end{aligned}
$$

By the argument above there are no generators a', b'', c'' such that $pa' = b''a'$, $pb' = c''b''$. So there are no generators a', b', c' satisfying (*), where a' is a scalar multiple of c.

Putting all this together, we see that 6.115 does not have a set of generators a', b', c' satisfying (*), and so 6.115 cannot be isomorphic to any of the algebras 6.114. Similar arguments show that 6.116 and 6.117 cannot be isomorphic to any of the algebras 6.114.

Next we show that the algebras 6.115 ˘ 6.117 are distinct. To this end we consider possible generating sets a', b', c' satisfying

$$pa' = b'a', \; pb' = c'a'. \tag{(**)}$$

Let L be the algebra 6.115. As above, we ørst consider the case when $a' = a$. To ensure that $pa' = b'a'$ we require $b' = b + \alpha a$ for some α, so that $pb' = \alpha ba + ca$. Let $c' = \lambda a + \mu b + \nu c$ (where necessarily $\nu \neq 0$). Then

$$
\begin{aligned}
c'a' &= \mu ba + \nu ca, \\
c'b' &= (\alpha\mu - \lambda) ba + \alpha\nu ca + \nu cb, \\
pc' &= \lambda ba + \mu ca + \nu cb
\end{aligned}
$$

To ensure that $pb' = c'a'$ we require $\mu = \alpha$, $\nu = 1$. Then

$$pc' = (2\lambda - \alpha^2) b'a' + c'b',$$

so that a', b', c' cannot satisfy the relations of 6.116 or 6.117. The same argument as above then shows that no generators a', b', c' for 6.115 can satisfy the relations of 6.116 or 6.117.

So 6.115 is not isomorphic to 6.116 or 6.117, and a similar argument shows that 6.116 is not isomorphic to 6.117.

Finally, we need to show that the p algebras 6.114 are distinct from each other. The proof of this is similar to the ones just given. We consider possible generating sets a', b', c' for 6.114 satisfying (*), and we show that diœerent parameters k in 6.114 give diœerent algebras.

4.2 Descendants of 4.2

Algebra 4.2 has no immediate descendants of order p^6.

4.3 Descendants of 4.3

If L is an immediate descendant pf 4.3 then L is generated by a, b, c, L_2 is generated modulo L_3 by ba, and L_3 is generated by baa, bab. Furthermore pa, pb, pc, ca, cb are all linear combinations of baa, bab. The commutator structure on L must be the same as that of one of 6.11 ˘ 6.15 from the list of Lie algebras over $\mathbb{Z}_p$ of dimension 6. So we may suppose that one of the following sets of commutator identities hold:

$$\begin{aligned} ca &= cb = 0, \\ cb &= 0,\ ca = bab, \\ cb &= 0,\ ca = baa, \\ ca &= bab,\ cb = \omega baa. \end{aligned}$$

4.3.1 Case 1

Assume that $ca = cb = 0$. If $pc = 0$ then $L = M \oplus \langle c \rangle$, where M must be isomorphic to one of 5.38 ˘ 5.46. So we have

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.118}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.119}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.120}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.121}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baa,\ pb - \lambda bab,\ pc,\ \text{class } 3\rangle, \tag{6.122}$$

with $\lambda \neq 0$ and λ, λ^{-1} giving isomorphic algebras ($(p+1)/2$ algebras),

$$\langle a, b, c \mid ca,\ cb,\ pa - baa - bab,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.123}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baa - \omega bab,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.124}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - \omega bab,\ pb - baa,\ pc,\ \text{class } 3\rangle, \tag{6.125}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - \alpha bab,\ pb - baa - bab,\ pc,\ \text{class } 3\rangle, \tag{6.126}$$

where $1 + 4\alpha$ is not a square ($(p-1)/2$ algebras).

On the other hand, if $pc \neq 0$ then we can assume that $pc = bab$, and subtracting suitable multiples of c from a, b we mays suppose that $pa = \alpha baa$, $pb = \beta baa$ for some α, β. If $\beta = 0$ then we can scale b and c so that $pa = 0$ or baa. On the other hand, if $\beta \neq 0$ then replacing a by $a - \alpha\beta^{-1}b$ we have $pa = 0$. Subtracting a suitable multiple of c from b again, we have $pb = \beta baa$, and then scaling a and c we can assume that $\beta = 1$ or ω. So we have

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc - bab,\ \text{class } 3\rangle, \tag{6.127}$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baa,\ pb,\ pc - bab,\ \text{class } 3\rangle, \tag{6.128}$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb - baa,\ pc - bab,\ \text{class } 3\rangle, \tag{6.129}$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb - \omega baa,\ pc - bab,\ \text{class } 3\rangle. \tag{6.130}$$

4.3.2 Case 2

Next suppose that $cb = 0$, $ca = bab$. Note that $B = \langle b, c\rangle + L_2$ and $C = \langle c\rangle + L_2$ are characteristic subalgebras.

If $pL = \{0\}$ then we have

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 3\rangle. \tag{6.131}$$

If $pB = \{0\}$, but $pa \neq 0$ then we consider the case when $pa \in B$ separately from the case $pa \notin B$. If $pa \in B$ then $pa = \lambda bab$ for some $\lambda \neq 0$. Scaling b and c we can assume that $\lambda = 1$ or ω. If $pa \notin B$ then we can assume that $pa = baa$. So we have

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.132}$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.133}$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle. \tag{6.134}$$

Algebra 6.134 cannot be isomorphic to 6.132 or 6.133 since $pL \leq B$ in 6.132 and 6.133, but not in 6.134. To see that 6.132 is not isomorphic to 6.133 we consider all possible generating sets a', b', c' for L satisfying the same commutator relations as 6.132 and 6.133, and such that $pB = \{0\}$, $pL \leq B$.

Next consider the case when $pC = \{0\}$ and $pB \neq \{0\}$. So $pc = 0$, and $pb \neq 0$. If $pb \in B$ then scaling a we may suppose that $pb = bab$. On the other hand, if $pb \notin B$ then we may assume that $pb = \lambda baa$, and scaling a we may suppose that $\lambda = 1$ or ω. Now let $pa = \alpha baa + \beta bab$.

Consider the case when $pb = bab$. We consider possible generating sets a', b', c' for L satisfying the same commutator relations as a, b, c and also satisfying the relations $pb' = b'a'b'$, $pc' = 0$. It is easy to see that

$$\begin{aligned} a' &= \lambda a + \mu b \text{ modulo } C, \\ b' &= \lambda^{-1} b \text{ modulo } C \end{aligned}$$

for some λ, μ with $\lambda \neq 0$. So

$$\begin{aligned} pa' &= \alpha\lambda baa + (\beta\lambda + \mu)bab, \\ pb' &= \lambda^{-1}bab, \\ b'a'a' &= \lambda baa + \mu bab, \\ b'a'b' &= \lambda^{-1}bab. \end{aligned}$$

So $pa' = \alpha b'a'a' + \lambda(\beta\lambda + \mu - \alpha\mu)b'a'b'$. If $\alpha \neq 1$ we can choose μ so that $pa' = \alpha b'a'a'$, and if $\alpha = 1$ then we can choose λ so that $pa' = b'a'a' + \beta' b'a'b'$ where $\beta' = 0, 1$, or ω. So we have p algebras

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \alpha baa,\ pb - bab,\ pc,\ \text{class } 3\rangle \ (0 \leq \alpha < p), \tag{6.135}$$

and

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - baa - bab,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.136}$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - baa - \omega bab,\ pb - bab,\ pc,\ \text{class } 3\rangle. \tag{6.137}$$

Next, consider the case when $pb = baa$. Once again, we consider generating sets a', b', c' for L satisfying the same commutator relations as a, b, c and also satisfying the relations $pb' = b'a'a'$, $pc' = 0$. It is easy to see that

$$\begin{aligned} a' &= \pm a \text{ modulo } C, \\ b' &= \lambda b \text{ modulo } C \end{aligned}$$

for some λ with $\lambda \neq 0$. So

$$\begin{aligned} pa' &= \pm(\alpha baa + \beta bab), \\ pb' &= \lambda baa, \\ b'a'a' &= \lambda baa, \\ b'a'b' &= \pm\lambda^2 bab. \end{aligned}$$

So

$$pa' = \pm\alpha\lambda^{-1}b'a'a' + \beta\lambda^{-2}b'a'b'.$$

Absorbing the $\pm$ sign into λ we can write

$$pa' = \alpha\lambda^{-1}b'a'a' + \beta\lambda^{-2}b'a'b'.$$

If $\alpha \neq 0$ then we can choose $\lambda = \alpha$, and this gives p algebras

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - baa - \beta bab,\ pb - baa,\ pc,\ \text{class } 3\rangle\ (0 \leq \beta < p). \quad (6.138)$$

On the other hand, if $\alpha = 0$ we can choose λ so that $\beta = 0, 1$, or ω. So we have

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - baa,\ pc,\ \text{class } 3\rangle, \quad (6.139)$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - bab,\ pb - baa,\ pc,\ \text{class } 3\rangle, \quad (6.140)$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - \omega bab,\ pb - baa,\ pc,\ \text{class } 3\rangle. \quad (6.141)$$

The case $pb = \omega baa$ is similar, and gives p algebras

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - baa - \beta bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle\ (0 \leq \beta < p), \quad (6.142)$$

and

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle, \quad (6.143)$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle, \quad (6.144)$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - \omega bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle. \quad (6.145)$$

Finally, consider the case when $pC \neq \{0\}$. If $pc \in B$ then scaling c we may suppose that $pc = bab$, and if $pc \notin B$ then we may assume that $pc = baa$.

First consider the case when $pc = bab$. Subtracting a suitable multiple of c from b we may suppose that $pb = \lambda baa$. If $\lambda = 0$ then subtacting a suitable multiple of c from a we may suppose that $pa = \mu baa$, and by scaling we can take $\mu = 0$ or 1. But if $\lambda \neq 0$ then we can subtract a suitable linear combination of b, c from a so that $pa = 0$. So we have p algebras

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - \lambda baa,\ pc - bab,\ \text{class } 3\rangle\ (0 \leq \lambda < p), \quad (6.146)$$

and

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - baa,\ pb,\ pc - bab,\ \text{class } 3\rangle. \quad (6.147)$$

Note that if $\lambda = 0$ in 6.146 then pL has order p, whereas pL has order p^2 in 6.147 and in 6.146 with $\lambda \neq 0$. Also pB has order p^2 in 6.146 with $\lambda \neq 0$, but pB has order p in 6.147. To see that we get diœerent algebras in 6.146 for diœerent non-zero values of λ we let L be 6.146 for some $\lambda \neq 0$, and we determine all possible generatings sets a', b', c' for L which satisfy the same commutator relations as a, b, c, and also satisfy the relations $pa' = 0$, $pc' = b'a'b'$. It is straightforward to check that

$$\begin{aligned} a' &= a \text{ modulo } L^2, \\ b' &= \beta b \text{ modulo } L^2, \\ c' &= \beta^2 c \text{ modulo } L^2 \end{aligned}$$

for some $\beta \neq 0$. So
$$pb' = \beta pb = \beta\lambda baa = \lambda b'a'a'.$$

And now consider the case when $pc = baa$. Subtracting suitable mulstiples of c from a and b we may suppose that $pa = \alpha bab$, $pb = \beta bab$ for some α, β. We consider all possible generating sets a', b', c' for L satisfying the same commutator relations as a, b, c, and also satisfying $pa' = \alpha' b'a'b'$, $pb' = \beta' b'a'b'$ for some α', β', $pc' = b'a'a'$. It is straightforward to show that

$$\begin{aligned}
a' &= \lambda a \text{ modulo } L^2,\\
b' &= \lambda^2 b \text{ modulo } L^2,\\
c' &= \lambda^4 c \text{ modulo } L^2
\end{aligned}$$

for some $\lambda \neq 0$. So

$$\begin{aligned}
pa' &= \lambda pa = \alpha\lambda^{-4} b'a'b',\\
pb' &= \lambda^2 pb = \beta\lambda^{-3} b'a'b'.
\end{aligned}$$

First consider the case when $\beta = 0$. If $p = 1 \bmod 4$ then we can choose λ so that $\alpha\lambda^{-4} = 0, 1, \omega, \omega^2$ or ω^3, and if $p = 3 \bmod 4$ then we can choose λ so that $\alpha\lambda^{-4} = 0, 1$ or ω. If $\beta \neq 0$ and $p = 1 \bmod 3$ then we can choose λ so that $\beta\lambda^{-3} = 1, \omega$ or ω^2, and if $p = 2 \bmod 3$ then we can choose λ so that $\beta\lambda^{-3} = 1$. This still leaves us the freedom to replace α by $\alpha\lambda^{-1}$ for any λ such that $\lambda^3 = 1$. So the number of algebras depends on the value of $p \bmod 12$.

$p \bmod 12$	algebras
1	$p+7$
5	$p+5$
7	$p+5$
11	$p+3$

When $\beta = 0$ and $p = 1 \bmod 4$ we have øve algebras

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \alpha bab,\ pb,\ pc - baa,\ \text{class } 3\rangle\ (\alpha = 0, 1, \omega, \omega^2, \omega^3), \quad (6.148)$$

and when $\beta = 0$ and $p = 3 \bmod 4$ we have three algebras

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \alpha bab,\ pb,\ pc - baa,\ \text{class } 3\rangle\ (\alpha = 0, 1, \omega), \quad (6.148\text{A})$$

When $\beta \neq 0$ and $p = 1 \bmod 3$ we have $p+2$ algebras

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \alpha bab,\ pb - \beta bab,\ pc - baa,\ \text{class } 3\rangle\ (\beta = 1, \omega, \omega^2), \quad (6.149)$$

where for a øxed value of β, two values α_1, α_2 of α give isomorphic algebras if and only if $\alpha_1^3 = \alpha_2^3$. And ønally, when $\beta \neq 0$ and $p = 2 \bmod 3$ we have p algebras

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \alpha bab,\ pb - bab,\ pc - baa,\ \text{class } 3\rangle\ (0 \leq \alpha < p). \quad (6.149\text{A})$$

4.3.3 Case 3

Next suppose that $cb = 0$, $ca = baa$. As above, note that $C = \langle c \rangle + L_2$ is a characteristic subalgebra.

If $pL = \{0\}$ then we have

$$\langle a, b, c \mid ca - baa,\ cb,\ pa,\ pb,\ pc,\ \text{class } 3\rangle. \tag{6.150}$$

Next suppose that $pC = \{0\}$, but that $pL \neq \{0\}$. Let

$$\begin{aligned} pa &= \lambda baa + \mu bab, \\ pb &= \nu baa + \xi bab. \end{aligned}$$

If a', b', c' generate L, and satsify the same commutator relations as a, b, c then it is straightforward to show that either

$$\begin{aligned} a' &= \alpha a \text{ modulo } C, \\ b' &= \beta b \text{ modulo } C, \\ c' &= \alpha\beta c \text{ modulo} L^3 \end{aligned} \tag{(*)}$$

or

$$\begin{aligned} a' &= \beta b \text{ modulo } C, \\ b' &= \alpha a \text{ modulo } C, \\ c' &= \alpha\beta c - \alpha\beta ba \text{ modulo } L^3 \end{aligned} \tag{(**)}$$

for some $\alpha, \beta \neq 0$. If a', b' satisfy (*) then

$$\begin{aligned} pa' &= \alpha pa = \alpha\lambda baa + \alpha\mu bab = \alpha^{-1}\beta^{-1}\lambda b'a'a' + \beta^{-2}\mu b'a'b', \\ pb' &= \beta pb = \beta\nu baa + \beta\xi bab = \alpha^{-2}\nu b'a'a' + \alpha^{-1}\beta^{-1}\xi b'a'b', \end{aligned}$$

and if a', b' satsify (**) then

$$\begin{aligned} pa' &= \beta pb = \beta\nu baa + \beta\xi bab = -\alpha^{-1}\beta^{-1}\xi b'a'a' - \alpha^{-2}\nu b'a'b' \\ pb' &= \alpha pa = \alpha\lambda baa + \alpha\mu bab = -\beta^{-2}\mu b'a'a' - \alpha^{-1}\beta^{-1}\lambda b'a'b'. \end{aligned}$$

Clearly, if one of pa, pb is zero, then we may assume that $pb = 0$. Then we can choose β in (*) so that $\beta^{-2}\mu = 0, 1$ or ω, and once β is øxed, then we can choose α so that $\alpha^{-1}\beta^{-1}\lambda = 0$ or 1. Omitting the case $\lambda = \mu = 0$ (which gives 6.150) this gives

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.151}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.152}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.153}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - \omega bab,\ pb,\ pc, \text{ class } 3\rangle, \tag{6.154}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb,\ pc, \text{ class } 3\rangle. \tag{6.155}$$

So assume that pa and pb are both non-zero. First consider the case when $\lambda = \xi = 0$. Then μ and ν are both non-zero, and using a change of generators of the form (*) we may assume that (μ, ν) is one of $(1,1)$, $(1,\omega)$, $(\omega,1)$, (ω,ω). If $p = 1 \bmod 4$ then -1 is a square, and using a change of generators of the form (**) we see that $(1,\omega)$ and $(\omega,1)$ give isomorphic algebras. But if $p = 3 \bmod 4$ then -1 is not a square, so that using a change of generators of the form (**) we see that $(1,1)$ and (ω,ω) give isomorphic algebras. So if $p = 1 \bmod 4$ we have

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb - baa,\ pc, \text{ class } 3\rangle, \tag{6.156}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb - \omega baa,\ pc, \text{ class } 3\rangle, \tag{6.157}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - \omega bab,\ pb - \omega baa,\ pc, \text{ class } 3\rangle. \tag{6.158}$$

And if $p = 3 \bmod 4$ we have

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb - baa,\ pc, \text{ class } 3\rangle, \tag{6.156A}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb - \omega baa,\ pc, \text{ class } 3\rangle, \tag{6.157A}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - \omega bab,\ pb - baa,\ pc, \text{ class } 3\rangle. \tag{6.158A}$$

Next consider the case when pa and pb are both non-zero, and one of λ, ξ is zero, and the other non-zero. Then we may assume that $\lambda \neq 0$, $\xi = 0$. This implies that $\nu \neq 0$, so we can choose α so that $\alpha^{-2}\nu = 1$ or ω, and then we can choose β so that $\alpha^{-1}\beta^{-1}\lambda = 1$. So we have $2p$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \mu bab,\ pb - baa,\ pc, \text{ class } 3\rangle\ (0 \leq \mu < p), \tag{6.159}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \mu bab,\ pb - \omega baa,\ pc, \text{ class } 3\rangle\ (0 \leq \mu < p). \tag{6.160}$$

If λ, ξ are both non-zero, but $\mu = \nu = 0$ then we can take $\lambda = 1$ with $(1, \xi)$ and $(1, \xi^{-1})$ giving isomorphic algebras. So we have $(p+1)/2$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa,\ pb - \xi bab,\ pc, \text{ class } 3\rangle\ (\xi \neq 0,\ \xi \sim \xi^{-1}). \tag{6.160}$$

If λ, ξ are both non-zero, and exactly one of μ, ν is non-zero, then we may suppose that $\mu \neq 0$, $\nu = 0$, and we can take $\lambda = 1$, $\mu = 1$ or ω. This gives $2(p-1)$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb - \xi bab,\ pc, \text{ class } 3\rangle\ (\xi \neq 0), \tag{6.161}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb - \xi bab,\ pc, \text{ class } 3\rangle\ (\xi \neq 0). \tag{6.162}$$

Finally consider the case when λ, μ, ν, ξ are all non-zero. Then we can take $\lambda = 1$ and let ξ take a value in S, where S is a set of representatives for the classes $\{\xi, \xi^{-1}\}$

of non-zero elements in $\mathbb{Z}_p$. We can also take $\mu = 1$ or ω. If $\xi \in S$ and $\xi \neq \pm 1$ then a change of generating set of the form (**) with $\alpha\beta = -\xi$ keeps $\lambda = 1$ but changes ξ to ξ^{-1} which does not lie in S. So for each $\xi \in S \backslash \{\pm 1\}$ we have $2(p-1)$ algebras with $\lambda = 1$, $\mu = 1$ or ω, and $\nu \neq 0$. This gives $(p-1)(p-3)$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb - \nu baa - \xi bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0,\ \xi \in S \backslash \{\pm 1\}), \tag{6.163}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb - \nu baa - \xi bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0,\ \xi \in S \backslash \{\pm 1\}), \tag{6.164}$$

If $\lambda = 1$, $\xi = \pm 1$ then a change of generating set of the form (**) with $\alpha\beta = -1$ øxes λ, ξ, and changes the pair (μ, ν) to $(-\alpha^{-2}\nu, -\alpha^2\mu)$. We consider four separate cases:

1. μ, ν both squares,
2. μ a square, ν not a square,
3. μ not a square, ν a square,
4. μ, ν both not squares.

If $p = 1 \bmod 4$ then -1 is a square and so a change of generating set of the form (**) preserves cases (1) and (4), but interchanges cases (2) and (3) so we obtain $3(p-1)$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb - \nu^2 baa \pm bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0), \tag{6.165}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb - \nu^2 \omega baa \pm bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0), \tag{6.166}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb - \nu^2 \omega baa \pm bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0). \tag{6.167}$$

Finally, if $p = 3 \bmod 4$ then -1 is not a square so a change of generating set of the form (**) interchanges cases (1) and (4), while preserving cases (2) and (3), so we obtain $3(p-1)$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb - \nu^2 baa \pm bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0), \tag{6.165A}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb - \nu^2 \omega baa \pm bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0), \tag{6.166A}$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb - \nu^2 baa \pm bab,\ pc, \text{ class } 3\rangle\ (\nu \neq 0). \tag{6.167A}$$

Now consider the case when $pc = \lambda baa + \mu bab \neq 0$. If we let a', b', c' satisfy (*) then

$$pc' = \alpha^{-1}\lambda b'a'a' + \beta^{-1}\mu b'a'b',$$

and if we let a', b', c' satisfy (**) then

$$pc' = -\beta^{-1}\mu b'a'a' - \alpha^{-1}\lambda b'a'b'.$$

So if one of λ, μ is non-zero and the other is zero we may assume that $\lambda = 1$ and $\mu = 0$, and if both λ, μ are non-zero then we may assume that $\lambda = \mu = 1$. Subtracting suitable multiples of c from a and b we may suppose that pa, pb are linear multiples of bab.

Suppose that $pc = baa$ and that $pa = \nu bab$, $pb = \xi bab$. If we consider possible generating sets a', b', c' for L satisfying the same commutator relations as a, b, c, and satisfying $pc' = b'a'a'$, $pa', pb' \in Sp\langle b'a'b'\rangle$ then it is straightofrward to check that

$$\begin{aligned} a' &= a \text{ modulo } L^2, \\ b' &= \beta b \text{ modulo } L^2, \\ c' &= \beta c \text{ modulo } L^3. \end{aligned}$$

So

$$\begin{aligned} pa' &= pa = \nu bab = \beta^{-2}\nu b'a'b', \\ pb' &= \beta pb = \beta\xi bab = \beta^{-1}\xi b'a'b'. \end{aligned}$$

If $\xi = 0$ then we can choose β so that $\beta^{-2}\nu = 0, 1$, or ω, giving

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa,\ pb,\ pc - baa,\ \text{class } 3\rangle, \tag{6.168}$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - bab,\ pb,\ pc - baa,\ \text{class } 3\rangle, \tag{6.169}$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab,\ pb,\ pc - baa,\ \text{class } 3\rangle. \tag{6.170}$$

And if $\xi \neq 0$ we can choose β so that $\beta^{-1}\xi = 1$, giving p algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \nu bab,\ pb - bab,\ pc - baa,\ \text{class } 3\rangle\ (0 \leq \nu < p). \tag{6.171}$$

Finally consider the case when $pc = baa + bab$, $pa = \nu bab$, $pb = \xi bab$. As above we consider possible generating sets a', b', c' for L satisfying the same commutator relations as a, b, c, and satisfying $pc' = b'a'a' + b'a'b'$, $pa', pb' \in Sp\langle b'a'b'\rangle$ then it is straightofrward to check that

$$\begin{aligned} a' &= a \text{ modulo } L^2, \\ b' &= b \text{ modulo } L^2, \\ c' &= c \text{ modulo } L^3, \end{aligned}$$

or

$$\begin{aligned} a' &= -b + \xi c \text{ modulo } L^2, \\ b' &= -a + \nu c \text{ modulo } L^2, \\ c' &= c - ba \text{ modulo } L^3. \end{aligned}$$

Clearly a change of generating set of the ørst of these two kinds øxes ν, ξ, but a change of generating set of the second of these two kinds swaps ν, ξ. So we have $p(p+1)/2$ algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \nu bab,\ pb - \xi bab,\ pc - baa - bab,\ \text{class } 3\rangle\ (0 \leq \nu \leq \xi < p). \tag{6.172}$$

4.3.4 Case 4

Finally, consider the case when $ca = bab$, $cb = \omega baa$. Once again, note that $C = \langle c\rangle + L_2$ is a characteristic subalgebra. It is straightforward to show that if a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } C, \\ b' &= \pm(\omega\beta a + \alpha b) \text{ modulo } C, \\ c' &= (\alpha^2 - \omega\beta^2)c \text{ modulo } L^3 \end{aligned}$$

for some α, β which are not both zero. (In establishing this fact, we first show that if $a' = a$ modulo C then $b' = \pm b$ modulo C and $c' = c$ modulo L^3. It follows from this that b' and c' are determined by a'.)

If $pL = \{0\}$ then we have

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc, \text{ class } 3\rangle. \tag{6.173}$$

Now consider the case when $pC = 0$, but $pL \neq \{0\}$. If pL has order p then we may suppose that $pa \neq 0$, $pb = 0$. It is straightforward to show that if a', b', c' generate L and satisfy the same commutator relations as a, b, c, and also satisfy $pb' = pc' = 0$ then

$$\begin{aligned} a' &= \alpha a \text{ modulo } C, \\ b' &= \pm\alpha b \text{ modulo } C, \\ c' &= \alpha^2 c \text{ modulo } L^3 \end{aligned}$$

for some $\alpha \neq 0$. So if $pa = \lambda baa + \mu bab$ then

$$pa' = \alpha pa = \alpha\lambda baa + \alpha\mu bab = \pm\alpha^{-2}\lambda b'a'a' + \alpha^{-2}\mu b'a'b'.$$

If $\mu \neq 0$ then we can take $\mu = 1$ or ω and we get $p + 1$ algebras

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - bab,\ pb,\ pc, \text{ class } 3\rangle\ (0 \leq \lambda \leq (p-1)/2), \tag{6.174}$$

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - \omega bab,\ pb,\ pc, \text{ class } 3\rangle\ (0 \leq \lambda \leq (p-1)/2), \tag{6.175}$$

with λ and $-\lambda$ giving isomorphic algebras. On the other hand if $\mu = 0$ we have to distinguish between the cases $p = 1 \bmod 4$ and $p = 3 \bmod 4$. If $p = 1 \bmod 4$ then -1 is a square, so we have two algebras

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - baa,\ pb,\ pc, \text{ class } 3\rangle, \tag{6.176}$$

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \omega baa,\ pb,\ pc, \text{ class } 3\rangle. \tag{6.177}$$

But if $p = 3 \bmod 4$ then -1 is not a square, so we have only one algebra

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - baa,\ pb,\ pc, \text{ class } 3\rangle. \tag{6.176A}$$

Next consider the case when $pC = \{0\}$ and pa, pb span a space of dimension two. We write

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

for some non-singular matrix A. We let

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } C, \\ b' &= \pm(\omega\beta a + \alpha b) \text{ modulo } C, \end{aligned}$$

and then

$$\begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix} = \pm(\alpha^2 - \omega\beta^2) \begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix} \begin{pmatrix} baa \\ bab \end{pmatrix}$$

and

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \frac{1}{\det P} PAP^{-1} \begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix}$$

where

$$P = \begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix}.$$

The set of all matrices P of this form form a group G of order $2(p^2 - 1)$, and the isomorphism classes of algebras L with $pC = \{0\}$ and pa, pb linearly independant correspond to the orbits of the set of all non-singular 2×2 matrices A under the action of G deøned by setting

$$A \to \frac{1}{\det P} PAP^{-1}.$$

We will show that there are $p^2 + (p-3)/2$ such orbits when $p = 1 \bmod 4$, and $p^2 + (p-1)/2$ such orbits when $p = 3 \bmod 4$. So we obtain

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - \mu bab,\ pb - \nu baa - \xi bab,\ pc, \text{ class } 3\rangle, \quad (6.178)$$

where $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ runs over a set of representatives for these orbits. [It may help in computing these orbit representatives to note that every orbit contains a representative with $\lambda = 0$ or $\lambda = 1$.]

We need to calculate the number of orbits of 2×2 matrices under the action of G. For each $P \in G$ we count the number of non-singular matrices A such that

$$\frac{1}{\det P} PAP^{-1} = A.$$

Clearly this number is zero unless $\det(P) = 1$ and $PAP^{-1} = A$, or $\det(P) = -1$ and $PAP^{-1} = -A$. Note that $\det(P) = \pm(\alpha^2 - \omega\beta^2)$. As (α, β) runs through all $p^2 - 1$ possibilities other than $(0, 0)$, $\alpha^2 - \omega\beta^2$ runs through all $p - 1$ non-zero elements in $\mathbb{Z}_p$. If $P = \begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$ and $Q = \begin{pmatrix} \gamma & \delta \\ \omega\delta & \gamma \end{pmatrix}$ then $\det P = \det Q$ if and only if $\det(PQ^{-1}) = 1$, and this implies that each possible value of $\alpha^2 - \omega\beta^2$ arises the same number of times as the value 1. So each possible value arises $p+1$ times. So there are $p+1$ pairs (α, β) with $\alpha^2 - \omega\beta^2 = 1$, and $p+1$ pairs with $\alpha^2 - \omega\beta^2 = -1$.

Now let $P = \begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$ with $\det P = 1$. We need to count how many non-singular 2×2 matrices are centralized by P. If $\beta = 0$ then $P = \pm I$, and P centralizes all $(p^2-1)(p^2-p)$ non-singular 2×2 matrices. If $\beta \neq 0$ then P only centralizes matrices of the form $\lambda I + \mu P$ and so P centralizes $p^2 - 1$ matrices.

Next let $P = \begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$ with $\det P = 1$. Here, again, P only centralizes matrices of the form $\lambda I + \mu P$, but we need to count the number of matrices of this form which are non-singular. Now

$$\det(\lambda I + \mu P) = \lambda^2 - \mu^2\alpha^2 + \mu^2\omega\beta^2 = \lambda^2 + \mu^2$$

since $\det P = 1$. So if $\mu = 0$ there are $p-1$ values of λ with $\det(\lambda I + \mu P) \neq 0$, and if $\mu \neq 0$ then there are either $p-2$ or p values of λ such that $\det(\lambda I + \mu P) \neq 0$, depending on whether $p = 1 \bmod 4$ or $p = 3 \bmod 4$. So P centralizes $(p-1)^2$ or $p^2 - 1$ matrices, depending on whether $p = 1 \bmod 4$ or $p = 3 \bmod 4$.

Next, we consider matrices P with $\det P = -1$, and for these we count non-singular matrices A such that $PAP^{-1} = -A$. If $P = \begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$ then it is straightforward to show that there are no such A. Finally, if $P = \begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$ and $\det P = -1$ then there are $(p-1)^2$ such A.

It is now a simple calculation to show that the number of orbits is $p^2 + (p-3)/2$ if $p = 1 \bmod 4$, and $p^2 + (p-1)/2$ if $p = 3 \bmod 4$.

Finally, consider the case when $pC \neq \{0\}$. We can then suppose that $pc = bab$, and subtracting suitable multiples of c from a and b we can suppose that $pa = \lambda baa$, $pb = \mu baa$ for some λ, μ. The only possible generating sets a', b', c' for L satisfying the same type of commutator and power relations as a, b, c are of the form

$$\begin{aligned} a' &= a \text{ modulo } C, \\ b' &= \pm b \text{ modulo } C, \\ c' &= c \text{ modulo } L^3. \end{aligned}$$

So we have $p(p+1)/2$ algebras

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa,\ pb - \mu baa,\ pc - bab,\ \text{class } 3 \rangle, \qquad (6.179)$$

where λ and $-\lambda$ give isomorphic algebras for any given μ, so that we get distinct algebras if we let $0 \leq \lambda \leq (p-1)/2$, $0 \leq \mu < p$.

4.4 Descendants of 4.4 and 4.5

Algebras 4.4 and 4.5 are terminal.

4.5 Descendants of 5.8

Let L be an immediate descendant of 5.8 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by pa, pb, and that L_3 is generated by p^2a. Furthermore we may suppose that $p^2b = pc = 0$, and we may suppose that ba, ca, cb are all scalar multiples of p^2a. So we may suppose that $cb = 0$ or p^2a. If $cb = p^2a$ then subtracting a suitable linear combination of b and c we may suppose that $ba = ca = 0$. This gives

$$\langle a, b, c \,|\, ba,\ ca,\ cb - p^2a,\ p^2b,\ pc,\ \text{class } 3\rangle. \tag{6.180}$$

If $cb = 0$ then we may assume that $ca = 0$ or p^2a. If $ca = p^2a$ then subtracting a suitable multiple of c from b we may suppose that $ba = 0$. This gives

$$\langle a, b, c \,|\, ba,\ ca - p^2a,\ cb,\ p^2b,\ pc,\ \text{class } 3\rangle. \tag{6.181}$$

Finally, if $ca = cb = 0$ we may suppose that $ba = 0$ or p^2a. So we have two more algebras

$$\langle a, b, c \,|\, ba,\ ca,\ cb,\ p^2b,\ pc,\ \text{class } 3\rangle. \tag{6.182}$$

$$\langle a, b, c \,|\, ba - p^2a,\ ca,\ cb,\ p^2b,\ pc,\ \text{class } 3\rangle. \tag{6.183}$$

These four algebras are distinct, since it is easy to see that if $p^2b = pc = 0$ and $p^2a \neq 0$ then the subalgebras $B = \langle b, c\rangle + L_2$ and $C = \langle c, pb\rangle + L_3$ are characteristic. Algebra 6.182 is abelian, but the other three algebras are non-abelian. In 6.182 C is central, but C is not central in 6.180 or 6.181. And ønally, B is abelian in 6.181, but not in 6.180.

4.6 Descendants of 5.9

Let L be an immediate descendant of 5.9 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by pa, ba, and that L_3 has order p and is generated by p^2a, baa, bab. The elements pb, pc, ca, cb all lie in L_3. Note that the subalgebras $B = \langle b, c\rangle + L_2$ and $C = \langle c\rangle + L_2$ are characteristic.

First consider the case when $L^3 = \{0\}$, so that $baa = bab = 0$, and L_3 is spanned by p^2a. Subtracting suitable multiples of pa from b and c we may suppose that $pb = pc = 0$. So $ca = \lambda p^2a$ and $cb = \mu p^2a$ for some λ, μ. One possibility is $\lambda = \mu = 0$. If $\mu \neq 0$ then subtracting a suitable multiple of b from a we may suppose that $\lambda = 0$, and by scaling we may suppose that $\mu = 1$. And if $\mu = 0$ and $\lambda \neq 0$ then by scaling we may assume that $\lambda = 1$. So we have three algebras

$$\langle a, b, c \,|\, baa,\ bab,\ ca,\ cb,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.184}$$

$$\langle a, b, c \mid baa,\ bab,\ ca - p^2a,\ cb,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.185}$$

$$\langle a, b, c \mid baa,\ bab,\ ca,\ cb - p^2a,\ pb,\ pc,\ \text{class } 3\rangle. \tag{6.186}$$

Note that in 6.184 the characteristic subalgebra C is central; in 6.185 C is not central, but $CB = \{0\}$; and in 6.186 $CB \neq \{0\}$.

Next suppose that $L^2B = \{0\}$, but that $L^3 \neq \{0\}$. Then $bab = 0$ but $baa \neq 0$. Scaling we may assume that $p^2a = 0$ or baa.

First consider the case when $p^2a = 0$. Then pb and pc are both scalar multiples of baa. If $pc = 0$ then scaling a we may suppose that $pb = 0$, baa or ωbaa. And if $pc \neq 0$ then scaling c we may suppose that $pc = baa$, and subtracting a suitable multiple of c from b we may suppose that $pb = 0$. So one of the following sets of equations holds:

$$\begin{aligned}
bab &= p^2a = pb = pc = 0, \\
bab &= p^2a = pc = 0,\ pb = baa, \\
bab &= p^2a = pc = 0,\ pb = \omega baa, \\
bab &= p^2a = pb = 0,\ pc = baa.
\end{aligned}$$

It is straightforward to see that algebras satisfying diœerent sets of relations from these four sets cannot be isomorphic. And if $p^2a \neq 0$ then by scaling we may suppose that $p^2a = baa$, and subtracting suitable multiples of pa from b and c we may suppose that $pb = pc = 0$. So we have

$$bab = pb = pc = 0,\ p^2a = baa.$$

For each of these sets of relations we have to consider the possibilities for ca, cb. By subtracting a suitable scalar multiple of ba from c we may suppose that $ca = 0$, and by scaling we may suppose that $cb = 0$ or baa. So we have

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.187}$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.188}$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb - baa,\ pc,\ \text{class } 3\rangle, \tag{6.189}$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb - baa,\ pc,\ \text{class } 3\rangle, \tag{6.190}$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle, \tag{6.191}$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle, \tag{6.192}$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb,\ pc - baa,\ \text{class } 3\rangle, \tag{6.193}$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb,\ pc - baa,\ \text{class } 3\rangle, \tag{6.194}$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a - baa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.195}$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a - baa,\ pb,\ pc,\ \text{class } 3\rangle. \tag{6.196}$$

Finally, consider the case when $L^2B \neq \{0\}$. Then L_3 is spanned by bab, and subtracting a suitable scalar multiple of b from a we may suppose that $baa = 0$. Scaling, we may assume that $p^2a = 0$ or bab or ωbab. If $p^2a \neq 0$ then subtracting suitable multiples of pa from b and c we may suppose that $pb = pc = 0$. If $p^2a = pc = 0$ but $pb \neq 0$ then we may assume that $pb = bab$. And if $p^2a = 0$ but $pc \neq 0$, then subtracting a suitable multiple of c from b we may assume that $pb = 0$, and scaling we may assume that $pc = bab$. Subtracting a suitable scalar multiple of ba from c we may suppose that $cb = 0$. And by scaling we may take $ca = 0$ or bab. So we have the following algebras:

$$\langle a, b, c \,|\, baa,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.197}$$

$$\langle a, b, c \,|\, baa,\ ca - bab,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.198}$$

$$\langle a, b, c \,|\, baa,\ ca,\ cb,\ p^2a,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.199}$$

$$\langle a, b, c \,|\, baa,\ ca - bab,\ cb,\ p^2a,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.200}$$

$$\langle a, b, c \,|\, baa,\ ca,\ cb,\ p^2a,\ pb,\ pc - bab,\ \text{class } 3\rangle, \tag{6.201}$$

$$\langle a, b, c \,|\, baa,\ ca - bab,\ cb,\ p^2a,\ pb,\ pc - bab,\ \text{class } 3\rangle, \tag{6.202}$$

$$\langle a, b, c \,|\, baa,\ ca,\ cb,\ p^2a - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.203}$$

$$\langle a, b, c \,|\, baa,\ ca - bab,\ cb,\ p^2a - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.204}$$

$$\langle a, b, c \,|\, baa,\ ca,\ cb,\ p^2a - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.205}$$

$$\langle a, b, c \,|\, baa,\ ca - bab,\ cb,\ p^2a - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle. \tag{6.206}$$

4.7 Descendants of 5.10

Let L be an immediate descendant of 5.10 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by pa, ba, and that L_3 has order p and is generated by p^2a, baa. The elements pc, ca, cb all lie in L_3, and $pb = ba$ modulo L_3. Furthermore, $bab = 0$. Note that the subalgebras $B = \langle b, c\rangle + L_2$ and $C = \langle c\rangle + L_2$ are characteristic.

One posibility is that $baa = 0$ and L_3 is spanned by p^2a. On the other hand, if $baa \neq 0$ then $p^2a = \lambda baa$ for some λ. But then setting $a' = a - \lambda b$ we have

$$p^2a' = p^2a - \lambda p^2b = p^2a - \lambda pba = p^2a - \lambda baa = 0.$$

So replacing a by a' we may suppose that $p^2a = 0$.

Suppose øIrst that $baa = 0$, and that L_3 is spanned by p^2a. Then subtracting suitable scalar multiples of pa from b and c we may suppose that $pb = ba$, $pc = 0$. If $cb = 0$ then scaling we may suppose that $ca = 0$ or p^2a. And if $cb \neq 0$ then we may suppose that $cb = p^2a$, and subtracting a suitable scalar multiple of b from a we may suppose that $ca = 0$. So we have three algebras:

$$\langle a, b, c \mid baa,\ ca,\ cb,\ pb - ba,\ pc,\ \text{class } 3\rangle, \tag{6.207}$$

$$\langle a, b, c \mid baa,\ ca - p^2a,\ cb,\ pb - ba,\ pc,\ \text{class } 3\rangle, \tag{6.208}$$

$$\langle a, b, c \mid baa,\ ca,\ cb - p^2a,\ pb - ba,\ pc,\ \text{class } 3\rangle. \tag{6.209}$$

Next, suppose that L_3 is spanned by baa and that $p^2a = 0$. Replacing a by $a + \alpha pa$ for suitable α, we may suppose that $pb = ba$. And replacing c by $c + \beta ba + \gamma pa$ for suitable β, γ, we may suppose that $ca = cb = 0$. Scaling, we may suppose that $pc = 0$ or baa

$$\langle a, b, c \mid ca,\ cb,\ p^2a,\ pb - ba,\ pc,\ \text{class } 3\rangle, \tag{6.210}$$

$$\langle a, b, c \mid ca,\ cb,\ p^2a,\ pb - ba,\ pc - baa,\ \text{class } 3\rangle. \tag{6.211}$$

4.8 Descendants of 5.11

Let L be an immediate descendant of 5.11 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by pa, ba, and that L_3 has order p and is generated by p^2a. The elements pb, ca, cb all lie in L_3, and $pc = ba$ modulo L_3. Note that the subalgebras $B = \langle b\rangle + L_2$ and $C = \langle c\rangle + L_2$ are characteristic. We also have $baa = bab = 0$.

Subtracting suitable scalar multiples of pa from b and c we can suppose that $pb = 0$, $pc = ba$. Note that if we replace a, b, c by a', b', c' where

$$\begin{aligned} a' &= \alpha a \text{ modulo } \langle b, c\rangle + L_2, \\ b' &= \beta b \text{ modulo } L_2, \\ c' &= \gamma c \text{ modulo } L_2, \end{aligned}$$

then we must have $\gamma = \alpha\beta$ to preserve the relation $pc = ba$. So by scaling we can assume that $cb = 0$ or p^2a or ωp^2a. And if $cb \neq 0$ then we can subtract a suitable scalar multiple of b from a so that $ca = 0$. Finally, if $cb = 0$ then we can assume that $ca = 0$ or p^2a. So we have

$$\langle a, b, c \mid ca,\ cb,\ pb,\ pc - ba,\ \text{class } 3\rangle, \tag{6.212}$$

$$\langle a, b, c \mid ca,\ cb - p^2a,\ pb,\ pc - ba,\ \text{class } 3\rangle, \tag{6.213}$$

$$\langle a, b, c \mid ca,\ cb - \omega p^2a,\ pb,\ pc - ba,\ \text{class } 3\rangle, \tag{6.214}$$

$$\langle a, b, c \mid ca - p^2a,\ cb,\ pb,\ pc - ba,\ \text{class } 3\rangle. \tag{6.215}$$

4.9 Descendants of 5.12

Let L be an immediate descendant of 5.12 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by pc, ba, and that L_3 has order p and is generated by baa, bab, p^2c. The elements pa, pb, ca, cb all lie in L_3. Note that the subalgebras $A = \langle a, b \rangle + L_2$ and $C = \langle c \rangle + L_2$ are characteristic.

First consider the case when $baa = bab = 0$, so that L_3 is spanned by p^2c. Subtracting suitable linear combinations of pc from a and b, we may suppose that $pa = pb = 0$. Since ca and cb are linearly independant we may suppose that $cb = 0$ and that $ca = 0$ or p^2c. So we have

$$\langle a, b, c \,|\, baa,\ bab,\ ca,\ cb,\ pa,\ pb,\ \text{class } 3 \rangle, \tag{6.216}$$

$$\langle a, b, c \,|\, baa,\ bab,\ ca - p^2c,\ cb,\ pa,\ pb,\ \text{class } 3 \rangle. \tag{6.217}$$

If ba is not central, we may suppose that $bab = 0$ and that L_3 is spanned by baa. Scaling we may suppose that $p^2c = 0$ or baa.

Consider the case when $bab = p^2c = 0$ and L_3 is spanned by baa. Note that $B = \langle b \rangle + L_2$ is characteristic. Subtracting a suitable scalar multiple of ba from c we may suppose that $ca = 0$. Scaling, we may suppose that $cb = 0$ or baa. If $pb = 0$ then we may suppose that $pa = 0$ or baa. And if $pb \neq 0$ then we can suppose that $pb = baa$ or ωbaa, and subtracting a suitable scalar multiple of b from a we may suppose that $pa = 0$. So we have

$$\langle a, b, c \,|\, bab,\ ca,\ cb,\ pa,\ pb,\ p^2c,\ \text{class } 3 \rangle, \tag{6.218}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb,\ pa - baa,\ pb,\ p^2c,\ \text{class } 3 \rangle, \tag{6.219}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb,\ pa,\ pb - baa,\ p^2c,\ \text{class } 3 \rangle, \tag{6.220}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb,\ pa,\ pb - \omega baa,\ p^2c,\ \text{class } 3 \rangle, \tag{6.221}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c,\ \text{class } 3 \rangle, \tag{6.222}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa - baa,\ pb,\ p^2c,\ \text{class } 3 \rangle, \tag{6.223}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - baa,\ p^2c,\ \text{class } 3 \rangle, \tag{6.224}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega baa,\ p^2c,\ \text{class } 3 \rangle. \tag{6.225}$$

Finally, consider the case when $bab = 0$, $p^2c = baa \neq 0$. Subtracting suitable scalar multiples of pc from a and b we may suppose that $pa = pb = 0$. And subtracting a suitable scalar multiple of ba from c we may suppose that $ca = 0$. Scaling, we may suppose that $cb = 0$ or baa. So we have

$$\langle a, b, c \,|\, bab,\ ca,\ cb,\ pa,\ pb,\ p^2c - baa,\ \text{class } 3 \rangle, \tag{6.226}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c - baa,\ \text{class } 3 \rangle. \tag{6.227}$$

4.10 Descendants of 5.13

Let L be an immediate descendant of 5.13 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by pc, ba, and that L_3 has order p and is generated by p^2c. We have $baa = bab = 0$ The elements pb, ca, cb all lie in L_3, and $pa = ba + \lambda p^2 c$ for some λ. Note that the subalgebras $A = \langle a, b\rangle + L_2$, $B = \langle b\rangle + L_2$ and $C = \langle c\rangle + L_2$ are characteristic.

Subtracting suitable scalar multiples of pc from a and b we may suppose that $pa = ba$, $pb = 0$. If $cb = 0$ then by scaling we may suppose that $ca = 0$ or p^2c. And if $cb \neq 0$ then subtracting a suitable scalar multiple of b from a we may suppose that $ca = 0$. So we have $ca = 0$, $cb = \lambda p^2 c$ for some $\lambda \neq 0$. This gives

$$\langle a, b, c \,|\, ca,\ cb,\ pa - ba,\ pb,\ \text{class } 3\rangle, \tag{6.228}$$

$$\langle a, b, c \,|\, ca - p^2c,\ cb,\ pa - ba,\ pb,\ \text{class } 3\rangle, \tag{6.229}$$

and $p-1$ algebras

$$\langle a, b, c \,|\, ca,\ cb - \lambda p^2 c,\ pa - ba,\ pb,\ \text{class } 3\rangle\ (\lambda \neq 0). \tag{6.230}$$

4.11 Descendants of 5.14

Let L be an immediate descendant of 5.14 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by ba, ca, and that L_3 has order p and is generated by commutators of weight 3. The elements pa, pb, pc, cb all lie in L_3. Note that the subalgebra $B = \langle b, c\rangle + L_2$ is characteristic. The commutator structure of L must be the same as in one of 6.16 ˘ 6.21 from the list of nilpotent Lie algebras of dimension 6 over $\mathbb{Z}_p$.

4.11.1 Case 1

Let L have the same commutator structure as 6.16, so that L^3 is spanned by bab, and

$$cb = baa = bac = caa = cab = cac = 0.$$

It is straightforward to check that $\langle a, c\rangle + L_2$, $\langle b, c\rangle + L_2$, $\langle c\rangle + L_2$ are all characteristic subalgebras. The elements pa, pb, pc must all be scalar multiples of bab. If $pc \neq 0$ then we may suppose that $pc = bab$, $pa = pb = 0$. If $pc = 0$ then we may suppose that $pa = \lambda bab$, $pb = \mu bab$ where $\lambda = 0, 1$ or ω, and $\mu = 0$ or 1. So we have

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.231}$$

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac,\ pa,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.232}$$

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.233}$$

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac,\ pa - bab,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.234}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.235}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa - \omega bab,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.236}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - bab,\ \text{class } 3\rangle. \tag{6.237}$$

4.11.2 Case 2

Let L have the same commutator structure as 6.17, so that L^3 is spanned by bab, and

$$cb = baa = bac = cab = cac = 0,\ caa = bab.$$

If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= -\alpha\delta b + \beta\delta c \text{ modulo } L_2, \\ c' &= \alpha\delta^2 c \text{ modulo } L_2. \end{aligned}$$

So we may suppose that $pc = 0$ or bab or ωbab. If $pc \neq 0$ then we may suppose that $pa = pb = 0$. If $pc = 0$ then we may suppose that $pb = 0$ or bab, and if $pb \neq 0$ then we can suppose that $pa = 0$. Finally, if $pb = pc = 0$ then we can assume that $pa = 0$ or bab or ωbab. So we have

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa - bab,\ cac,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.238}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa - bab,\ cac,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.239}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa - bab,\ cac,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.240}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa - bab,\ cac,\ pa,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.241}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa - bab,\ cac,\ pa,\ pb,\ pc - bab,\ \text{class } 3\rangle, \tag{6.242}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa - bab,\ cac,\ pa,\ pb,\ pc - \omega bab,\ \text{class } 3\rangle. \tag{6.243}$$

4.11.3 Case 3

Let L have the same commutator structure as 6.18, so that L^3 is spanned by bab, and

$$cb = baa = bac = caa = cab = 0,\ cac = -bab.$$

If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a \text{ modulo } L_2, \\ b' &= \beta b + \gamma c \text{ modulo } L_2, \\ c' &= \pm(\gamma b + \beta c) \text{ modulo } L_2, \end{aligned}$$

where $\beta^2 - \gamma^2 \neq 0$. Note that if a', b', c' are as above, then $b'a'b' = \alpha(\beta^2 - \gamma^2)bab$.

If $pb = pc = 0$ then since we can choose β, γ so that $\beta^2 - \gamma^2$ takes any non-zero value, we may take $pa = 0$ or bab. This gives

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac + bab,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.244}$$

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac + bab,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle. \tag{6.245}$$

If one of pb, pc is zero, and the other is non-zero, then we may assume that $pb \neq 0$, $pc = 0$. Then with a change of generators of the above form, with $\gamma = 0$, we can choose β so that $pa' = 0$ or $b'a'b'$ or $\omega b'a'b'$, and then we can choose α so that $pb' = b'a'b'$. So we have

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac + bab,\ pa,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.246}$$

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac + bab,\ pa - bab,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.247}$$

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac + bab,\ pa - \omega bab,\ pb - bab,\ pc,\ \text{class } 3\rangle. \tag{6.248}$$

Now consider the case when pb and pc are both non-zero. We can scale a so that $pb = bab$, $pc = \lambda bab$ for some $\lambda \neq 0$. So we have $p(\lambda b - c) = 0$. Provided $\lambda \neq \pm 1$ we can replace a, b, c by a, b', c' where

$$\begin{aligned} b' &= b + \lambda c, \\ c' &= \lambda b - c, \end{aligned}$$

and then $pc = 0$ and we are back in the case above. If $\lambda = \pm 1$, then replacing c by $-c$ if necessary, we may assume that $\lambda = 1$. As above, we can choose β, γ so that $\beta^2 - \gamma^2$ takes any non-zero value, and so we may assume that $pa = 0$ or bab. So we have two extra algebras

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac + bab,\ pa,\ pb - bab,\ pc - bab,\ \text{class } 3\rangle, \tag{6.249}$$

$$\langle a, b, c \,|\, cb,\ baa,\ bac,\ caa,\ cac + bab,\ pa - bab,\ pb - bab,\ pc - bab,\ \text{class } 3\rangle. \tag{6.250}$$

4.11.4 Case 4

Let L have the same commutator structure as 6.19, so that L^3 is spanned by bab, and

$$cb = baa = bac = caa = cab = 0,\ cac = -\omega bab.$$

If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a \text{ modulo } L_2, \\ b' &= \beta b + \gamma c \text{ modulo } L_2, \\ c' &= \pm(\omega\gamma b + \beta c) \text{ modulo } L_2, \end{aligned}$$

where $\beta^2 - \omega\gamma^2 \neq 0$. Note that if a', b', c' are as above, then $b'a'b' = \alpha(\beta^2 - \omega\gamma^2)bab$.

The analysis of the situation when at least one of pb, pc equals 0 is identical to the case above. So we have the following algebras:

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac + \omega bab,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.251}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac + \omega bab,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.252}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac + \omega bab,\ pa,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.253}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac + \omega bab,\ pa - bab,\ pb - bab,\ pc,\ \text{class } 3\rangle, \tag{6.254}$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac + \omega bab,\ pa - \omega bab,\ pb - bab,\ pc,\ \text{class } 3\rangle. \tag{6.255}$$

Now, as in the case above, consider the situation when $pb = bab$, $pc = \omega\lambda bab$ for some $\lambda \neq 0$. So we have $p(\omega\lambda b - c) = 0$. Since $1 - \omega\lambda^2 \neq 0$ we can replace a, b, c by a, b', c' where

$$\begin{aligned} b' &= -b + \lambda c, \\ c' &= \omega\lambda b - c, \end{aligned}$$

and we are back to the case when $pc = 0$.

4.11.5 Case 5

Let L have the same commutator structure as 6.20, so that L^3 is spanned by baa, and

$$cb = bab = bac = caa = cab = cac = 0.$$

The subalgebras $B = \langle b, c\rangle + L_2$ and $C = \langle c\rangle + L_2$ are characteristic. We may assume that $pc = 0$ or baa, and if $pc = baa$ thenw e can assume that $pa = pb = 0$. If $pc = 0$ then we can assume that $pb = 0$ or baa or ωbaa, and if $pb \neq 0$ we can assume that $pa = 0$. Finally, if $pb = pc = 0$ then we can assume that $pa = 0$ or baa. So we have

$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.256}$$

$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.257}$$

$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - baa,\ pc,\ \text{class } 3\rangle, \tag{6.258}$$

$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle, \tag{6.259}$$

$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - baa,\ \text{class } 3\rangle. \tag{6.260}$$

4.11.6 Case 6

Let L have the same commutator structure as 6.21, so that L^3 is spanned by baa, and

$$bab = bac = caa = cab = cac = 0,\ cb = baa.$$

The subalgebras $B = \langle b, c\rangle + L_2$ and $C = \langle c\rangle + L_2$ are characteristic. The analysis of this situation is exactly the same as the case above, and we have

$$\langle a, b, c \mid cb - baa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.261}$$

$$\langle a, b, c \mid cb - baa,\ bab,\ bac,\ caa,\ cac,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.262}$$

$$\langle a, b, c \mid cb - baa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - baa,\ pc,\ \text{class } 3\rangle, \tag{6.263}$$

$$\langle a, b, c \mid cb - baa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle, \tag{6.264}$$

$$\langle a, b, c \mid cb - baa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - baa,\ \text{class } 3\rangle. \tag{6.265}$$

4.12 Descendants of 5.15

Let L be an immediate descendant of 5.15 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by ba, ca, and that L_3 has order p and is generated by caa, cac. The elements pb, pc, cb all lie in L_3, and $pa = ba$ modulo L_3. We also have $baa = bab = bac = 0$. Note that the subalgebras $\langle b\rangle + L_2$ and $\langle b, c\rangle + L_2$ are characteristic.

If $cac \neq 0$ then we may suppose that $caa = cb = 0$. If $pb \neq 0$ then we can assume that $pb = cac$ and that $pa = ba$, $pc = 0$. This gives

$$\langle a, b, c \mid cb,\ caa,\ pa - ba,\ pb - cac,\ pc,\ \text{class } 3\rangle. \tag{6.266}$$

On the other hand, if $pb = 0$ we may assume that $pa = ba + \lambda cac$ where $\lambda = 0, 1$, or ω, and we may assume that $pc = 0$ or cac. So we have

$$\langle a, b, c \mid cb,\ caa,\ pa - ba,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.267}$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba,\ pb,\ pc - cac,\ \text{class } 3\rangle, \tag{6.268}$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - cac,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.269}$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - cac,\ pb,\ pc - cac,\ \text{class } 3\rangle, \tag{6.270}$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - \omega cac,\ pb,\ pc,\ \text{class } 3\rangle, \tag{6.271}$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - \omega cac,\ pb,\ pc - cac,\ \text{class } 3\rangle. \tag{6.272}$$

So suppose that $cac = 0$, and L_3 is spanned by caa. We have $cb, pb, pc \in L_3$, and $pa = ba$ modulo L_3. If a', b', c' generate L and satisfy relations of this form then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= b \text{ modulo } L_2, \\ c' &= \delta b + \varepsilon c \text{ modulo } L_2, \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon$. We may assume that $cb = 0, caa$ or ωcaa. And if $pb \neq 0$ we may suppose that $pb = caa$ and that $pa = ba$, $pc = 0$. So suppose that $pb = 0$. If $pa = ba + \lambda caa$ then replacing b by $b + \lambda ca$ we have $pa = ba$. We then have $pc = \mu caa$, where if $cb = 0$ then we may assume that $\mu = 0, 1$, or ω. So we have

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb - caa,\ pc,\ \text{class } 3 \rangle, \tag{6.273}$$

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc,\ \text{class } 3 \rangle, \tag{6.274}$$

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - caa,\ \text{class } 3 \rangle, \tag{6.275}$$

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - \omega caa,\ \text{class } 3 \rangle, \tag{6.276}$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb - caa,\ pc,\ \text{class } 3 \rangle, \tag{6.277}$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb,\ pc - \mu caa,\ \text{class } 3 \rangle \ (0 \leq \mu < p), \tag{6.278}$$

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb - caa,\ pc,\ \text{class } 3 \rangle, \tag{6.279}$$

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb,\ pc - \mu caa,\ \text{class } 3 \rangle \ (0 \leq \mu < p). \tag{6.280}$$

4.13 Descendants of 5.16

Let L be an immediate descendant of 5.16 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by ba, ca, and that L_3 has order p and is generated by caa, cac. The elements pa, pc, cb all lie in L_3, and $pb = ba$ modulo L_3. We also have $baa = bab = bac = 0$. Note that the subalgebras $\langle a, c \rangle + L_2$, $\langle b \rangle + L_2$ and $\langle c \rangle + L_2$ are characteristic.

If $cac \neq 0$ then subtracting a suitable scalar multiple of c from a we may suppose that $caa = 0$. And subtracting a suitable scalar multiple of ca from b we may suppose that $cb = 0$. We then have $pa = \lambda cac$, $pb = ba + \mu cac$, $pc = \nu cac$ for some λ, μ, ν. Scaling c we can take $\nu = 0$ or 1, and if $\nu = 0$ then we can scale c so that $\lambda = 0, 1$ or ω. Then, scaling b we can take $\mu = 0$ or 1. So we have

$$\langle a, b, c \,|\, cb,\ caa,\ pa,\ pb - ba,\ pc,\ \text{class } 3 \rangle, \tag{6.281}$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa,\ pb - ba - cac,\ pc,\ \text{class } 3 \rangle, \tag{6.282}$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - cac,\ pb - ba,\ pc,\ \text{class } 3 \rangle, \tag{6.283}$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - cac,\ pb - ba - cac,\ pc,\ \text{class } 3 \rangle, \tag{6.284}$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - \omega cac,\ pb - ba,\ pc,\ \text{class } 3 \rangle, \tag{6.285}$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - \omega cac,\ pb - ba - cac,\ pc,\ \text{class } 3 \rangle, \tag{6.286}$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - \lambda cac,\ pb - ba,\ pc - cac,\ \text{class } 3 \rangle \ (0 \leq \lambda < p), \tag{6.287}$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - \lambda cac,\ pb - ba - cac,\ pc - cac,\ \text{class } 3 \rangle \ (0 \leq \lambda < p). \tag{6.288}$$

If $cac = 0$ then L_3 is spanned by caa. Replacing b by $b + \alpha ca$ for suitable α we may suppose that $pb = ba$. And if $pc \neq 0$ then, subtracting a suitable scalar multiple of c from a, we may suppose that $pa = 0$. On the other hand, if $pc = 0$ then we scaling c we may suppose that $pa = 0$ or caa. Also, scaling b we may suppose that $cb = 0$ or caa. So we have

$$\langle a, b, c \mid cb,\ cac,\ pa,\ pb - ba,\ pc - \lambda caa, \text{ class } 3\rangle \ (0 \leq \lambda < p), \tag{6.289}$$

$$\langle a, b, c \mid cb - caa,\ cac,\ pa,\ pb - ba,\ pc - \lambda caa, \text{ class } 3\rangle \ (0 \leq \lambda < p), \tag{6.290}$$

$$\langle a, b, c \mid cb,\ cac,\ pa - caa,\ pb - ba,\ pc, \text{ class } 3\rangle, \tag{6.291}$$

$$\langle a, b, c \mid cb - caa,\ cac,\ pa - caa,\ pb - ba,\ pc, \text{ class } 3\rangle. \tag{6.292}$$

4.14 Descendants of 5.17

Algebra 5.17 is terminal.

4.15 Descendants of 5.18

Let L be an immediate descendant of 5.18 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by ba, ca, and that L_3 has order p and is generated by baa, bab. The elements pa, pc, cb all lie in L_3, and $pb = ca$ modulo L_3. We also have $bac = caa = cab = cac = 0$. If a', b', c' generate L and satisfy relations of this form, then

$$\begin{aligned} a' &= \alpha a + \gamma c \text{ modulo } L_2, \\ b' &= \alpha\varepsilon b + \delta c \text{ modulo } L_2, \\ c' &= \varepsilon c \text{ modulo } L_2 \end{aligned} \tag{(*)}$$

for some $\alpha, \gamma, \delta, \varepsilon$. The commutators baa and bab must be linearly dependant, and we may suppose that they satisfy one of the following three relations:

$$\begin{aligned} baa &= 0, \\ bab &= 0, \\ baa &= bab. \end{aligned}$$

And if $pc \neq 0$, then subtracting suitable scalar multiples of c from a and b we may assume that $pa = 0$, $pb = ca$.

4.15.1 $baa = 0$

Since $bab \neq 0$, by subtracting a suitable scalar multiple of ba from c we may suppose that $cb = 0$. If $pc \neq 0$ then we may assume that $pc = bab$, and we have

$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca,\ pc - bab,\ \text{class } 3\rangle. \tag{6.293}$$

So suppose that $pc = 0$, and that $pa = \lambda bab$, $pb = ca + \mu bab$. Then if let a', b', c' satisfy (*) we have

$$\begin{aligned} pa' &= \alpha\lambda bab = \alpha^{-2}\varepsilon^{-2}\lambda b'a'b', \\ pb' &= \alpha\varepsilon pb = c'a' + \alpha^{-2}\varepsilon^{-1}\mu b'a'b'. \end{aligned}$$

If $\mu \neq 0$ then we can take $\varepsilon = \alpha^{-2}\mu$ (so that $pb' = c'a' + b'a'b'$), and then choose α so that $\alpha^{-2}\varepsilon^{-2}\lambda = 0, 1$ or ω. And if $\mu = 0$ then we can take $\varepsilon = 1$ and again choose α so that $\alpha^{-2}\varepsilon^{-2}\lambda = 0, 1$ or ω. So we have

$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca,\ pc,\ \text{class } 3\rangle, \tag{6.294}$$

$$\langle a, b, c \mid cb,\ baa,\ pa - bab,\ pb - ca,\ pc,\ \text{class } 3\rangle, \tag{6.295}$$

$$\langle a, b, c \mid cb,\ baa,\ pa - \omega bab,\ pb - ca,\ pc,\ \text{class } 3\rangle, \tag{6.296}$$

$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca - bab,\ pc,\ \text{class } 3\rangle, \tag{6.297}$$

$$\langle a, b, c \mid cb,\ baa,\ pa - bab,\ pb - ca - bab,\ pc,\ \text{class } 3\rangle, \tag{6.298}$$

$$\langle a, b, c \mid cb,\ baa,\ pa - \omega bab,\ pb - ca - bab,\ pc,\ \text{class } 3\rangle. \tag{6.299}$$

4.15.2 $baa = bab$

As above, by subtracting a suitable scalar multiple of ba from c we may suppose that $cb = 0$. However, when we consider generating sets a', b', c' for L satisfying (*), then to preserve the relation $b'a'a' = b'a'b'$ we require $\varepsilon = 1$. As above, if $pc \neq 0$ then we can assume that $pa = 0$, $pb = ca$, and we can choose α so that $pc = baa$ when $p = 2 \bmod 3$, or $baa, \omega baa$ or $\omega^2 baa$ when $p = 1 \bmod 3$. This gives

$$\langle a, b, c \mid cb,\ bab - baa,\ pa,\ pb - ca,\ pc - baa,\ \text{class } 3\rangle, \tag{6.300}$$

and two extra algebras when $p = 1 \bmod 3$.

$$\langle a, b, c \mid cb,\ bab - baa,\ pa,\ pb - ca,\ pc - \omega baa,\ \text{class } 3\rangle, \tag{6.301}$$

$$\langle a, b, c \mid cb,\ bab - baa,\ pa,\ pb - ca,\ pc - \omega^2 baa,\ \text{class } 3\rangle. \tag{6.302}$$

So suppose that $pc = 0$, and that $pa = \lambda baa$, $pb = ca + \mu baa$. Then if we let a', b', c' satisfy (*) with $\varepsilon = 1$, we have

$$\begin{aligned} pa' &= \alpha\lambda baa = \alpha^{-2}\lambda b'a'a', \\ pb' &= \alpha ca + \alpha\mu baa = c'a' + \alpha^{-2}\mu b'a'a' \end{aligned}$$

We can choose α so that $\alpha^{-2}\lambda = 0, 1$ or ω, and if $\lambda = 0$ then we can choose α so that $\mu = 0, 1, \omega$.

$$\langle a, b, c \mid cb,\ bab - baa,\ pa,\ pb - ca,\ pc,\ \text{class } 3\rangle, \tag{6.303}$$

$$\langle a, b, c \mid cb,\ bab - baa,\ pa,\ pb - ca - baa,\ pc,\ \text{class } 3\rangle, \tag{6.304}$$

$$\langle a, b, c \mid cb,\ bab - baa,\ pa,\ pb - ca - \omega baa,\ pc,\ \text{class } 3\rangle, \tag{6.305}$$

$$\langle a, b, c \mid cb,\ bab - baa,\ pa - baa,\ pb - ca - \mu baa,\ pc,\ \text{class } 3\rangle\ (0 \le \mu < p), \tag{6.305B}$$

$$\langle a, b, c \mid cb,\ bab - baa,\ pa - \omega baa,\ pb - ca - \mu baa,\ pc,\ \text{class } 3\rangle\ (0 \le \mu < p). \tag{6.305C}$$

4.15.3 $bab = 0$

Since L_3 is spanned by baa, by subtracting a suitable multiple of ba from c we may assume that $pb = ca$. And if $pc \neq 0$ then we may assume that $pa = 0$. Let $cb = \lambda baa$, $pc = \mu baa$ with $\mu \neq 0$. Then if a', b', c' generate L and satisfy (*) we have

$$\begin{aligned} c'b' &= \alpha\varepsilon^2 cb = \alpha^{-2}\varepsilon\lambda b'a'a', \\ pc' &= \varepsilon pc = \alpha^{-3}\mu b'a'a'. \end{aligned}$$

So we have

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc - baa,\ \text{class } 3\rangle, \tag{6.306}$$

and two more algebras when $p = 1 \bmod 3$

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc - \omega baa,\ \text{class } 3\rangle, \tag{6.307}$$

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc - \omega^2 baa,\ \text{class } 3\rangle, \tag{6.308}$$

$$\langle a, b, c \mid cb - baa,\ bab,\ pa,\ pb - ca,\ pc - baa,\ \text{class } 3\rangle, \tag{6.309}$$

and two more algebras when $p = 1 \bmod 3$

$$\langle a, b, c \mid cb - baa,\ bab,\ pa,\ pb - ca,\ pc - \omega baa,\ \text{class } 3\rangle, \tag{6.310}$$

$$\langle a, b, c \mid cb - baa,\ bab,\ pa,\ pb - ca,\ pc - \omega^2 baa,\ \text{class } 3\rangle. \tag{6.311}$$

So let $pc = 0$, and let $cb = \lambda baa$, $pa = \mu baa$. If a', b', c' generate L and satisfy (*) we have

$$\begin{aligned} c'b' &= \alpha\varepsilon^2 cb = \alpha^{-2}\varepsilon\lambda b'a'a', \\ pa' &= \alpha pa = \alpha^{-2}\varepsilon^{-1}\mu b'a'a'. \end{aligned}$$

So if $\mu = 0$ then we can take $\lambda = 0$ or 1. And if $\mu \neq 0$ we can take $\mu = 1$, $\lambda = 0, 1$ or ω when $p = 3 \bmod 4$, and $\mu = 1$, $\lambda = 0, 1, \omega, \omega^2$ or ω^3 when $p = 1 \bmod 4$. So we have

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc,\ \text{class } 3\rangle, \tag{6.312}$$

$$\langle a,b,c \mid cb-baa,\ bab,\ pa,\ pb-ca,\ pc,\ \text{class }3\rangle, \tag{6.313}$$

$$\langle a,b,c \mid cb,\ bab,\ pa-baa,\ pb-ca,\ pc,\ \text{class }3\rangle, \tag{6.314}$$

$$\langle a,b,c \mid cb-baa,\ bab,\ pa-baa,\ pb-ca,\ pc,\ \text{class }3\rangle, \tag{6.315}$$

$$\langle a,b,c \mid cb-\omega baa,\ bab,\ pa-baa,\ pb-ca,\ pc,\ \text{class }3\rangle, \tag{6.316}$$

and in the case when $p=1 \bmod 4$ we also have two more algebras

$$\langle a,b,c \mid cb-\omega^2 baa,\ bab,\ pa-baa,\ pb-ca,\ pc,\ \text{class }3\rangle, \tag{6.317}$$

$$\langle a,b,c \mid cb-\omega^3 baa,\ bab,\ pa-baa,\ pb-ca,\ pc,\ \text{class }3\rangle. \tag{6.318}$$

4.16 Descendants of 5.19

Let L be an immediate descendant of 5.19 of order p^6. Then we may suppose that L is generated by a,b,c, that L_2 is generated modulo L_3 by ba, ca, and that L_3 has order p and is generated by bab. The elements pc and cb lie in L_3, and $pa=ba$ modulo L_3, $pb=ca$ modulo L_3. We also have $caa=-bab$, $baa=bac=cab=cac=0$. If a',b',c' generate L and satisfy relations of this form, then

$$\begin{aligned} a' &= \alpha a+\alpha\beta b+\gamma c \text{ modulo } L_2, \\ b' &= b+\beta c \text{ modulo } L_2, \\ c' &= \alpha^{-1}c \text{ modulo } L_2 \end{aligned} \tag{(*)}$$

for some α,β,γ. Note that $pba=-bab$. So subtracting a suitable scalar multiple of ba from c we may suppose that $pc=0$. And subtracting a suitable scalar multiple of ca from c, we may suppose that $pb=ca$. Also, subtracting a suitable scalar multiple of ca from b, we may suppose that $pa=ba$. Finally, we have $cb=\lambda bab$ for some λ. If we let $a'=\alpha a$, $b'=b$, $c'=\alpha^{-1}c$ then

$$c'b'=\alpha^{-1}cb=\alpha^{-1}\lambda bab=\alpha^{-2}\lambda b'a'b'.$$

So we have

$$\langle a,b,c \mid cb,\ pa-ba,\ pb-ca,\ pc,\ \text{class }3\rangle, \tag{6.319}$$

$$\langle a,b,c \mid cb-bab,\ pa-ba,\ pb-ca,\ pc,\ \text{class }3\rangle, \tag{6.320}$$

$$\langle a,b,c \mid cb-\omega bab,\ pa-ba,\ pb-ca,\ pc,\ \text{class }3\rangle. \tag{6.321}$$

4.17 Descendants of 5.20˘5.23

These algebras are all terminal.

4.18 Descendants of 5.24

Let L be an immediate descendant of 5.24 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by pa, that L_3 is generated modulo L_4 by p^2a, and that L_4 has order p and is generated by p^3a. The elements pb, pc, ba, ca, cb are all scalar multiples of p^3a. Subtracting suitable scalar multiples of p^2a from b and c we may suppose that $pb = pc = 0$. Note that these relations imply that $B = \langle b, c\rangle + L_4$ is a characteristic subalgebra. Clearly a is centralized by some non-zero element in the span of b, c, and so we may suppose that $ca = 0$. Scaling b we may suppose that $ba = 0$ or p^3a, and scaling c we may suppose that $cb = 0$ or p^3a. However, if $cb \neq 0$ then we can subtract a suitable scalar multiple of c from a so that $ba = 0$. So we have

$$\langle a, b, c \mid ba,\ ca,\ cb,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.322}$$

$$\langle a, b, c \mid ba - p^3a,\ ca,\ cb,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.323}$$

$$\langle a, b, c \mid ba,\ ca,\ cb - p^3a,\ pb,\ pc,\ \text{class } 4\rangle. \tag{6.324}$$

4.19 Descendants of 5.25˘5.26

These algebras are both terminal.

4.20 Descendants of 5.27

Let L be an immediate descendant of 5.27 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by ba, that L_3 is generated modulo L_4 by baa, and that L_4 has order p and is generated by $baaa$. The elements pa, pb, pc, bab, ca, cb are all scalar multiples of $baaa$. By subtracting a suitable scalar multiple of baa from c we may suppose that $ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, then

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2,\\
b' &= \delta b + \varepsilon c \text{ modulo } L_2,\\
c' &= \zeta c \text{ modulo } L_3
\end{aligned}$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. So if $pc \neq 0$ we may assume that $pa = pb = 0$, and if $pc = 0$, $pb \neq 0$ then we may assume that $pa = 0$.

Consider the case when $pa = pb = pc = 0$, and $bab = \lambda baaa$, $cb = \mu baaa$. Scaling b we can take $\lambda = 0$ or 1, and scaling c we can take $\mu = 0$ or 1. So we have

$$\langle a, b, c \mid bab,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.325}$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.326}$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.327}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb-baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle. \tag{6.328}$$

Next consider the case when $pb = pc = 0$, $pa \neq 0$. Then we may assume that $pa = baaa$. Once again we have $bab = \lambda baaa$, $cb = \mu baaa$ for some λ, μ. As above, we can scale c so that $\mu = 0$ or 1. And we can scale a and b so that $\lambda = 0, 1$ or ω, or (in the case when $p = 1 \bmod 4$) ω^2 or ω^3. So we have

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.329}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.330}$$

$$\langle a,b,c \mid bab-\omega baaa,\ ca,\ cb,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.331}$$

$$\langle a,b,c \mid bab-\omega^2 baaa,\ ca,\ cb,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4), \tag{6.332}$$

$$\langle a,b,c \mid bab-\omega^3 baaa,\ ca,\ cb,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4), \tag{6.333}$$

$$\langle a,b,c \mid bab,\ ca,\ cb-baaa,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.334}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb-baaa,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.335}$$

$$\langle a,b,c \mid bab-\omega baaa,\ ca,\ cb-baaa,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.336}$$

$$\langle a,b,c \mid bab-\omega^2 baaa,\ ca,\ cb-baaa,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4), \tag{6.337}$$

$$\langle a,b,c \mid bab-\omega^3 baaa,\ ca,\ cb-baaa,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4). \tag{6.338}$$

Now suppose that $pa = pc = 0$, $pb \neq 0$. Scaling a we may suppose that $pb = baaa$, or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. Scaling b and c we may suppose that bab and cb are both 0 or $baaa$. So we have

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa,\ pb-baaa,\ pc,\ \text{class } 4\rangle, \tag{6.339}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb,\ pa,\ pb-baaa,\ pc,\ \text{class } 4\rangle, \tag{6.340}$$

$$\langle a,b,c \mid bab,\ ca,\ cb-baaa,\ pa,\ pb-baaa,\ pc,\ \text{class } 4\rangle, \tag{6.341}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb-baaa,\ pa,\ pb-baaa,\ pc,\ \text{class } 4\rangle, \tag{6.342}$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.343}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.344}$$

$$\langle a,b,c \mid bab,\ ca,\ cb-baaa,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.345}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb-baaa,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.346}$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.347}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.348}$$

$$\langle a,b,c \mid bab,\ ca,\ cb-baaa,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.349}$$

$$\langle a,b,c \mid bab-baaa,\ ca,\ cb-baaa,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3), \tag{6.350}$$

Finally, consider the case when $pa = pb = 0$, and $pc \neq 0$. Then scaling c we may suppose that $pc = baaa$. Let $bab = \lambda baaa$, $cb = \mu baaa$. If we let $a' = \alpha a$, $b' = \beta b$, $c' = \alpha^3 \beta c$ then

$$\begin{aligned} pc' &= b'a'a'a', \\ b'a'b' &= \alpha\beta^2 bab = \alpha^{-2}\beta\lambda b'a'a'a', \\ c'b' &= \alpha^3\beta^2 cb = \beta\mu b'a'a'a'. \end{aligned}$$

So, if $\lambda = 0$ we can take $\mu = 0$ or 1, and if $\lambda \neq 0$ we can take $\lambda = 1$ and $\mu = 0, 1$ or ω. So we have

$$\langle a, b, c \mid bab,\ ca,\ cb,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle, \tag{6.351}$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle, \tag{6.352}$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle, \tag{6.353}$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle, \tag{6.354}$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb - \omega baaa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle. \tag{6.355}$$

4.21 Descendants of 5.28˘5.31

These algebras are all terminal.

4.22 Descendants of 5.32

Let L be an immediate descendant of 5.32 of order p^6. Then we may suppose that L is generated by a, b, c, that L_2 is generated modulo L_3 by ba, that L_3 is generated modulo L_4 by baa, and that L_4 has order p and is generated by $baaa$. The elements pa, pb, pc, bab, ca are all scalar multiples of $baaa$ and $cb = baa$ modulo L_4. We also have $bac = -baaa$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \delta b + \varepsilon c \text{ modulo } L_2, \\ c' &= \alpha^2 c - \alpha\beta ba \text{ modulo } L_3 \end{aligned} \tag{(*)}$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon$. Since $bac = -baaa$ we can choose ε so that $b'a'b' = 0$. So we can assume that $bab = 0$, and then if a', b', c' generate L and satisfy (*), and if in addition $b'a'b' = 0$ then $\varepsilon = 0$. If we let $a' = a + \gamma c$, $b' = b$, $c' = c$ then

$$\begin{aligned} b'a' &= ba - \gamma cb, \\ b'a'a' &= baa - 2\gamma baaa, \\ b'a'a'a' &= baaa, \end{aligned}$$

so we can choose γ so that $c'b' = b'a'a'$. So we can assume that $cb = baa$. Note that if a', b', c' generate L and satisfy (*), and if in addition $b'a'b' = 0$ and $c'b' = b'a'a'$ then $\varepsilon = \gamma = 0$. Also, subtracting a suitable multiple of baa from c we may assume that $ca = 0$. Next, note that if $pb \neq 0$ then we can subtract a suitable scalar multiple of b from a so that $pa = 0$.

So suppose that $pa = 0$, and that $pb \neq 0$. Scaling a we may suppose that $pb = baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. And we can scale b so that $pc = 0$ or $baaa$. So we have

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle, \tag{6.356}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3), \tag{6.357}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3), \tag{6.358}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - baaa,\ pc - baaa,\ \text{class } 4\rangle, \tag{6.359}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega baaa,\ pc - baaa,\ \text{class } 4\rangle \ (p = 1 \bmod 3), \tag{6.360}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaa,\ pc - baaa,\ \text{class } 4\rangle \ (p = 1 \bmod 3). \tag{6.361}$$

And now consider the case when $pb = 0$. Let $pa = \lambda baaa$, $pc = \mu baaa$. If we let $a' = \alpha a$, $b' = \delta b$, $c' = \alpha^2 c$ then

$$\begin{aligned} pa' &= \alpha pa = \alpha\lambda baaa = \alpha^{-2}\beta^{-1}\lambda b'a'a'a', \\ pc' &= \alpha^2 pc = \alpha^2\mu baaa = \alpha^{-1}\beta^{-1}\mu b'a'a'a'. \end{aligned}$$

So we may assume that $\lambda = 0$ or 1 and that $\mu = 0$ or 1. So we have

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.362}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle, \tag{6.363}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle, \tag{6.364}$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa - baaa,\ pb,\ pc - baaa,\ \text{class } 4\rangle. \tag{6.365}$$

4.23 Descendants of 5.33 ˘ 5.36

These algebras are all terminal.

5 Two generators

Let L be a two generator nilpotent Lie algebra of order p^6. Then L is an immediate descendant of one of 3.2 ˘ 3.4 or 4.6 ˘ 4.14 or 5.37 ˘ 5.74.

5.1 Descendants of 3.2 ˘ 3.4

Algebras 3.2 and 3.4 have no immediate descendants of order p^6, and 3.3 is terminal.

5.2 Descendants of 4.6

Let L be an immediate descendant of 4.6 of order p^6. Then L is generated by a, b modulo L_2, L_2 is generated by pa, pb modulo L_3, and L_3 has order p^2 and is generated by p^2a, p^2b. We have $ba \in L_3$, and so we may suppose that $ba = 0$ or p^2a. This gives

$$\langle a, b \,|\, ba, \text{ class } 3\rangle, \tag{6.366}$$

$$\langle a, b \,|\, ba - p^2a, \text{ class } 3\rangle. \tag{6.367}$$

5.3 Descendants of 4.7

Let L be an immediate descendant of 4.7 of order p^6. Then L is generated by a, b modulo L_2, L_2 is generated by ba, pa modulo L_3, and L_3 has order p^2 and is generated by baa, bab, p^2a. We also have $pb \in L_3$. Note that $\langle b\rangle + L_2$ is a characteristic subalgebra.

5.3.1 $L^3 = L_3$

If $L^3 = L_3$ then L_3 is spanned by baa, bab. If p^2a is a scalar multiple of bab then we can assume that $p^2a = 0$ or bab or ωbab. On the other hand, if p^2a is not a scalar multiple of bab then $p^2a = \alpha baa + \beta bab$ for some $\alpha \neq 0$. Scaling b we may assume that $\alpha = 1$, and then replacing a by $a + \beta b$ we have $p^2a = baa$.

If $p^2a = 0$, then in a similar way we may assume that $pb = 0$ or bab or baa or ωbaa. If $p^2a = bab$ or ωbab then subtracting a suitable multiple of pa from b we may suppose that $pb = \lambda baa$ where $\lambda = 0, 1$ or ω. And ønally, if $p^2a = baa$ then we may suppose that $pb = \lambda bab$ for some $0 \leq \lambda < p$. So we have

$$\langle a, b \,|\, p^2a, pb, \text{ class } 3\rangle, \tag{6.368}$$

$$\langle a, b \,|\, p^2a, pb - bab, \text{ class } 3\rangle, \tag{6.369}$$

$$\langle a, b \,|\, p^2a, pb - baa, \text{ class } 3\rangle, \tag{6.370}$$

$$\langle a, b \,|\, p^2a, pb - \omega baa, \text{ class } 3\rangle, \tag{6.371}$$

$$\langle a, b \,|\, p^2a - bab, pb, \text{ class } 3\rangle, \tag{6.372}$$

$$\langle a, b \,|\, p^2a - bab, pb - baa, \text{ class } 3\rangle, \tag{6.373}$$

$$\langle a, b \,|\, p^2a - bab, pb - \omega baa, \text{ class } 3\rangle, \tag{6.374}$$

$$\langle a, b \,|\, p^2a - \omega bab, pb, \text{ class } 3\rangle, \tag{6.375}$$

$$\langle a, b \,|\, p^2a - \omega bab,\ pb - baa,\ \text{class } 3\rangle, \tag{6.376}$$

$$\langle a, b \,|\, p^2a - \omega bab,\ pb - \omega baa,\ \text{class } 3\rangle, \tag{6.377}$$

$$\langle a, b \,|\, p^2a - baa,\ pb - \lambda bab,\ \text{class } 3\rangle\ (0 \leq \lambda < p). \tag{6.378}$$

5.3.2 $L^3 \neq L_3$

If $L^3 \neq L_3$ then baa and bab must be linearly dependant. We may assume that either $baa = 0$ or $bab = 0$.

If $baa = 0$, then L_3 is generated by bab and p^2a. Subtracting a suitable scalar multiple of pa from b we may suppose that $pb = \lambda bab$ where $\lambda = 0$ or 1:

$$\langle a, b \,|\, baa,\ pb,\ \text{class } 3\rangle, \tag{6.379}$$

$$\langle a, b \,|\, baa,\ pb - bab,\ \text{class } 3\rangle. \tag{6.380}$$

And if $bab = 0$ then we may similarly suppose that $pb = \lambda baa$ where $\lambda = 0, 1$ or ω:

$$\langle a, b \,|\, bab,\ pb,\ \text{class } 3\rangle, \tag{6.381}$$

$$\langle a, b \,|\, bab,\ pb - baa,\ \text{class } 3\rangle, \tag{6.382}$$

$$\langle a, b \,|\, bab,\ pb - \omega baa,\ \text{class } 3\rangle. \tag{6.383}$$

5.4 Descendants of 4.8

Let L be an immediate descendant of 4.8 of order p^6. Then L is generated by a, b modulo L_2, L_2 is generated by ba, pa modulo L_3, and L_3 has order p^2 and is generated by baa, p^2a. We also have $pb - ba \in L_3$. Note that $\langle b\rangle + L_2$ is a characteristic subalgebra.

Let $pb = ba + \lambda baa + \mu p^2a$. If we let $a' = a + \lambda pa$, $b' = b - \mu pa$ then

$$b'a' = ba + \lambda baa$$

and

$$pb' = pb - \mu p^2a = ba + \lambda baa = b'a'.$$

So we have

$$\langle a, b \,|\ pb - ba,\ \text{class } 3\rangle. \tag{6.384}$$

5.5 Descendants of 4.9 ˘ 4.14

Algebras 4.9 and 4.13 have no immediate descendants of order p^6, and 4.10 ˘ 4.12, 4.14 are terminal.

5.6 Descendants of 5.37

Let L be an immediate descendant of 5.37 order p^6. Then L is generated by a, b, L_2 is generated by ba, pa, pb modulo L_3, and L_3 has order p and is generated by $baa, bab, pba, p^2a, p^2b$.

First consider the case when $baa = bab = 0$. If $p^2a = p^2b = 0$ then we have

$$\langle a, b \mid baa,\ bab,\ p^2a,\ p^2b,\ \text{class } 3\rangle. \tag{6.385}$$

If at least one of p^2a and p^2b is non-zero then we may suppose that L_3 is generated by p^2a and that $p^2b = 0$. We can then assume that $pba = 0$ or p^2a. So we have

$$\langle a, b \mid baa,\ bab,\ pba,\ p^2b,\ \text{class } 3\rangle, \tag{6.386}$$

$$\langle a, b \mid baa,\ bab,\ p^2a - pba,\ p^2b,\ \text{class } 3\rangle. \tag{6.387}$$

Now consider the case when baa and bab are not both zero. We may assume that $bab = 0$ and that L_3 is generated by baa. We may assume that $pba = 0$ or baa.

First consider the case when $pba = 0$. If $p^2b \neq 0$ then we may assume that $p^2a = 0$ and that $p^2b = baa$ or ωbaa. And if $p^2b = 0$ then we may assume that $p^2a = 0$ or baa. This gives

$$\langle a, b \mid bab, pba,\ p^2a,\ p^2b,\ \text{class } 3\rangle, \tag{6.388}$$

$$\langle a, b \mid bab, pba,\ p^2a - baa,\ p^2b,\ \text{class } 3\rangle, \tag{6.389}$$

$$\langle a, b \mid bab, pba,\ p^2a,\ p^2b - baa,\ \text{class } 3\rangle, \tag{6.390}$$

$$\langle a, b \mid bab, pba,\ p^2a,\ p^2b - \omega baa,\ \text{class } 3\rangle. \tag{6.391}$$

Finally, consider the case when $bab = 0$, $pba = baa$. If $p^2b = 0$ then we can assume that $p^2a = 0$ or baa, and if $p^2b \neq 0$ then we can assume that $p^2a = 0$. So we have

$$\langle a, b \mid bab, pba - baa,\ p^2a,\ p^2b - \lambda baa,\ \text{class } 3\rangle\ (0 \leq \lambda < p), \tag{6.392}$$

$$\langle a, b \mid bab, pba - baa,\ p^2a - baa,\ p^2b,\ \text{class } 3\rangle. \tag{6.393}$$

5.7 Descendants of 5.38

Let L be an immediate descendant of 5.38 order p^6. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baaa$, $baab$, and $babb$. We also have $pa, pb \in L_4$.

From the list of nilpotent Lie algebras of dimension 6 over $\mathbb{Z}_p$, we see that we can assume that one of the following sets of relations holds:

$$\begin{aligned} baab &= babb = 0, \\ baab &= 0,\ babb = -baaa, \\ baab &= 0,\ babb = -\omega baaa. \end{aligned}$$

5.7.1 Case 1

First suppose that $baab = babb = 0$, and that L_3 is generated by $baaa$. Then the subalgebra $\langle b\rangle + L_2$ is characteristic. If $pb = 0$, we can assume that $pa = 0$ or $baaa$. And if $pb \neq 0$ then we may assume that $pa = 0$ and that $pb = baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. This gives

$$\langle a, b \mid baab, babb,\ pa,\ pb,\ \text{class } 4\rangle, \tag{6.394}$$

$$\langle a, b \mid baab, babb,\ pa - baaa,\ pb,\ \text{class } 4\rangle, \tag{6.395}$$

$$\langle a, b \mid baab, babb,\ pa,\ pb - baaa,\ \text{class } 4\rangle, \tag{6.396}$$

$$\langle a, b \mid baab, babb,\ pa,\ pb - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.397}$$

$$\langle a, b \mid baab, babb,\ pa,\ pb - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3). \tag{6.398}$$

5.7.2 Case 2

Next suppose that $baab = 0$ and that $babb = -baaa$. It is straightforward to show that if a', b' generate L and if $b'a'a'b' = 0$, $b'a'b'b' = -b'a'a'a'$ then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \pm(\beta a + \alpha b) \text{ modulo } L_2, \end{aligned}$$

for some α, β with $\alpha \neq \pm\beta$. If $pb = 0$ then we may assume that $pa = 0$ or $baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, giving

$$\langle a, b \mid baab, babb + baaa,\ pa,\ pb,\ \text{class } 4\rangle, \tag{6.399}$$

$$\langle a, b \mid baab, babb + baaa,\ pa - baaa,\ pb,\ \text{class } 4\rangle, \tag{6.400}$$

$$\langle a, b \mid baab, babb + baaa,\ pa - \omega baaa,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.401}$$

$$\langle a, b \mid baab, babb + baaa,\ pa - \omega^2 baaa,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3). \tag{6.402}$$

But if $pb \neq 0$ then we must have $pa = \lambda pb$ for some λ. Provided $\lambda \neq \pm 1$ we can take

$$\begin{aligned} a' &= -\lambda a + b, \\ b' &= a - \lambda b, \end{aligned}$$

and then we have $pb' = 0$, and replacing a, b by a', b' we are back in the case above.

So it remains to consider the case when $pa = \pm pb \neq 0$. Replacing b by $-b$ if necessary we may assume that $pa = pb = \mu baaa$ for some $\mu \neq 0$. If we let

$$\begin{aligned} a' &= (1 + \beta)a + \beta b, \\ b' &= \beta a + (1 + \beta)b \end{aligned}$$

for some $\beta \neq -1/2$, then

$$\begin{aligned} pa' &= pb' = (1+2\beta)\mu baaa, \\ b'a'a'a' &= (1+2\beta)^2 baaa. \end{aligned}$$

So if we choose β so that $1 + 2\beta = \mu$ then $pa' = pb' = b'a'a'a'$. So replacing a, b by a', b' we have

$$\langle a, b \mid baab, babb + baaa,\ pa - baaa,\ pb - baaa, \text{ class } 4\rangle. \tag{6.403}$$

5.7.3 Case 3

Finally consider the case when $baab = 0$ and $babb = -\omega baaa$. It is straightforward to show that if a', b' generate L and if $b'a'a'b' = 0$, $b'a'b'b' = -\omega b'a'a'a'$ then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \pm(\omega\beta a + \alpha b) \text{ modulo } L_2, \end{aligned}$$

for some α, β not both zero. We can always choose b' so that $pb' = 0$, and so we have

$$\langle a, b \mid baab, babb + \omega baaa,\ pa,\ pb, \text{ class } 4\rangle, \tag{6.404}$$

$$\langle a, b \mid baab, babb + \omega baaa,\ pa - baaa,\ pb, \text{ class } 4\rangle, \tag{6.405}$$

$$\langle a, b \mid baab, babb + \omega baaa,\ pa - \omega baaa,\ pb, \text{ class } 4\rangle\ (p = 1 \bmod 3), \tag{6.406}$$

$$\langle a, b \mid baab, babb + \omega baaa,\ pa - \omega^2 baaa,\ pb, \text{ class } 4\rangle\ (p = 1 \bmod 3). \tag{6.407}$$

5.8 Descendants of 5.39

Let L be an immediate descendant of 5.39 order p^6. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baaa$. We also have $pa - bab$ and $pb \in L_4$. Note that $\langle b\rangle + L_2$ is a characteristic subalgebra.

If $pb \neq 0$ then we may suppose that $pa = bab$. And scaling a we may suppose that $pb = baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. This gives

$$\langle a, b \mid pa - bab,\ pb - baaa, \text{ class } 4\rangle, \tag{6.408}$$

$$\langle a, b \mid pa - bab,\ pb - \omega baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3), \tag{6.409}$$

$$\langle a, b \mid pa - bab,\ pb - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3). \tag{6.410}$$

So suppose that $pb = 0$ and that $pa = bab + \lambda baaa$. If we let $a' = \alpha a$, $b' = \pm b$ then

$$pa' = b'a'b' \pm \alpha^{-2}\lambda b'a'a'a'.$$

So we have

$$\langle a, b \mid pa - bab,\ pb, \text{ class } 4\rangle, \tag{6.411}$$

$$\langle a, b \mid pa - bab - baaa,\ pb, \text{ class } 4\rangle, \tag{6.412}$$

$$\langle a, b \mid pa - bab - \omega baaa,\ pb, \text{ class } 4\rangle\ (p = 1 \bmod 4). \tag{6.413}$$

5.9 Descendants of 5.40

Let L be an immediate descendant of 5.40 order p^6. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baaa$. We also have $pa - \omega bab$ and $pb \in L_4$. Note that $\langle b \rangle + L_2$ is a characteristic subalgebra. This case is very similar to the case of the descendants of 5.39.

If $pb \neq 0$ then we may suppose that $pa = bab$. And scaling a we may suppose that $pb = baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. This gives

$$\langle a, b \mid pa - \omega bab,\ pb - baaa,\ \text{class } 4\rangle, \tag{6.414}$$

$$\langle a, b \mid pa - \omega bab,\ pb - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.415}$$

$$\langle a, b \mid pa - \omega bab,\ pb - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3). \tag{6.416}$$

So suppose that $pb = 0$ and that $pa = \omega bab + \lambda baaa$. If we let $a' = \alpha a$, $b' = \pm b$ then

$$pa' = \omega b'a'b' \pm \alpha^{-2}\lambda b'a'a'a'.$$

So we have

$$\langle a, b \mid pa - \omega bab,\ pb,\ \text{class } 4\rangle, \tag{6.417}$$

$$\langle a, b \mid pa - \omega bab - baaa,\ pb,\ \text{class } 4\rangle, \tag{6.418}$$

$$\langle a, b \mid pa - \omega bab - \omega baaa,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 4). \tag{6.419}$$

5.10 Descendants of 5.41

Let L be an immediate descendant of 5.41 order p^6. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $babb$. We also have $pa - baa$ and $pb \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_4, then

$$\begin{aligned} a' &= \alpha a \text{ modulo } L_2, \\ b' &= \alpha^{-1} b \text{ modulo } L_2 \end{aligned}$$

for some $\alpha \neq 0$.

Let $pa = baa + \lambda babb$, and let $pb = \mu babb$. We can assume that $\mu = 0$ or 1, and if $\mu = 0$ we can assume that $\lambda = 0$ or 1 or (in the case when $p = 1 \bmod 3$) ω or ω^2. So we have

$$\langle a, b \mid pa - baa,\ pb,\ \text{class } 4\rangle, \tag{6.420}$$

$$\langle a, b \mid pa - baa - babb,\ pb,\ \text{class } 4\rangle, \tag{6.421}$$

$$\langle a, b \mid pa - baa - \omega babb,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.422}$$

$$\langle a, b \mid pa - baa - \omega^2 babb,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.423}$$

$$\langle a, b \mid pa - baa - \lambda babb,\ pb - babb,\ \text{class } 4\rangle\ (0 \leq \lambda < p). \tag{6.424}$$

5.11 Descendants of 5.42

Algebra 5.42 has a presentation

$$\langle a, b \mid pa - baa,\ pb - \lambda bab,\ \text{class } 3\rangle,$$

where $\lambda \neq 0$. However this algebra is terminal unless $\lambda = -1$. So let L be an immediate descendant of 5.42 order p^6 with $\lambda = -1$. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baab$. We also have $pa - baa$ and $pb + bab \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_4, then

$$\begin{aligned} a' &= \alpha a \text{ modulo } L_2, \\ b' &= \alpha^{-1} b \text{ modulo } L_2 \end{aligned}$$

for some $\alpha \neq 0$, or

$$\begin{aligned} a' &= \alpha b \text{ modulo } L_2, \\ b' &= \alpha^{-1} a \text{ modulo } L_2 \end{aligned}$$

for some $\alpha \neq 0$.

Let $pa = baa + \mu baab$, $pb = -bab + \nu baab$. If one of μ, ν is non-zero, then swapping a and b if necessary we may assume that $\mu \neq 0$. Then scaling we may suppose that $\mu = 1$. So we have

$$\langle a, b \mid pa - baa,\ pb + bab,\ \text{class } 4\rangle, \tag{6.425}$$

$$\langle a, b \mid pa - baa - baab,\ pb + bab - \nu baab,\ \text{class } 4\rangle\ (0 \leq \nu < p). \tag{6.426}$$

5.12 Descendants of 5.43 ˘ 5.44

Algebras 5.43 and 5.44 are both terminal.

5.13 Descendants of 5.45

Let L be an immediate descendant of 5.45 order p^6. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baaa$, with $babb = -\omega baaa$. We also have $pa + bab$ and $pb + \omega baa \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_4, then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \pm(\omega\beta a + \alpha b) \text{ modulo } L_2 \end{aligned}$$

for some α, β with $\alpha^2 - \omega\beta^2 = 1$.

Let $pa + bab = \lambda baaa$ and $pb + \omega baa = \mu baaa$. If we let $a' = \alpha a + \beta b$ and $b' = \pm(\omega\beta a + \alpha b)$ then

$$\begin{aligned}
b'a' &= \pm ba, \\
b'a'a' &= \pm(\alpha baa + \beta bab), \\
b'a'b' &= \omega\beta baa + \alpha bab, \\
b'a'a'a' &= \pm baaa, \\
pa' + b'a'b' &= \pm(\alpha\lambda + \beta\mu)b'a'a'a', \\
pb' + \omega b'a'a' &= (\omega\beta\lambda + \alpha\mu)b'a'a'a'.
\end{aligned}$$

We deøne an equivalence relation on the set of ordered pairs $(\lambda, \mu) \in \mathbb{Z}_p \times \mathbb{Z}_p$ by setting

$$(\lambda', \mu') \sim (\lambda, \mu)$$

if

$$(\lambda', \mu') = (\pm(\alpha\lambda + \beta\mu), \omega\beta\lambda + \alpha\mu)$$

for some α, β with $\alpha^2 - \omega\beta^2 = 1$. So equivalent pairs correspond to isomorpic algebras. It is straightforward to show that if $(\lambda', \mu') \sim (\lambda, \mu)$ then $\mu'^2 - \omega\lambda'^2 = \mu^2 - \omega\lambda^2$. Conversely, if $(\lambda, \mu) \neq (0, 0)$ and $(\lambda', \mu') \neq (0, 0)$ then we can ønd α, β so that

$$(\lambda', \mu') = (\alpha\lambda + \beta\mu, \omega\beta\lambda + \alpha\mu),$$

and $\alpha^2 - \omega\beta^2 = 1$ if and only if $\mu'^2 - \omega\lambda'^2 = \mu^2 - \omega\lambda^2$. So there are p equivalence classes corresponding to the p distinct values of $\mu^2 - \omega\lambda^2$.Thus we have p algebras

$$\langle a, b \mid pa + bab - \lambda baaa,\ pb + \omega baa - \mu baaa,\ \text{class } 4\rangle \ (0 \leq \lambda, \mu < p), \qquad (6.427)$$

with the isomorphism class depending only on the value of $\mu^2 - \omega\lambda^2$.

If $p = 1 \bmod 4$ (so that -1 is a square) then a complete set of representatives of the p equivalence classes is given by

$$\{(\lambda, 0) \mid 0 \leq \lambda \leq (p-1)/2\} \cup \{(0, \mu) \mid 0 < \mu \leq (p-1)/2\}.$$

If $p = 3 \bmod 4$, then

$$\{(\lambda, 0) \mid 0 \leq \lambda \leq (p-1)/2\}$$

is a set of representatives for the equivalence classes with $\mu^2 - \omega\lambda^2 = k^2$ for some k. However it is not so easy to ønd a set of representatives for the classes with $\mu^2 - \omega\lambda^2$ not a square!

5.14 Descendants of 5.46

Algebra 5.46 is terminal.

5.15 Descendants of 5.47

Let L be an immediate descendant of 5.47 order p^6. Then L is generated by a, b, L_2 is generated by pa, pb modulo L_3, and L_3 is generated by p^2a modulo L_4, and L_4 has order p and is generated by p^3a. We also have ba and $p^2b \in L_4$. Subtracting a suitable scalar multiple of pa from b we may assume that $p^2b = 0$. And we may assume that $ba = 0$ or p^3a. So we have

$$\langle a, b \mid ba,\ p^2b,\ \text{class } 4\rangle, \tag{6.428}$$

$$\langle a, b \mid ba - p^3a,\ p^2b,\ \text{class } 4\rangle. \tag{6.429}$$

5.16 Descendants of 5.48

Let L be an immediate descendant of 5.48 order p^6. Then L is generated by a, b, L_2 is generated by pa, pb modulo L_3, and L_3 is generated by p^2a modulo L_4, and L_4 has order p and is generated by p^3a. We also have $ba - p^2a$ and $p^2b \in L_4$. Subtracting suitable scalar multiples of pa and pb from b we may assume that $ba = p^2a$ and that $p^2b = 0$.

$$\langle a, b \mid ba - p^2a,\ p^2b,\ \text{class } 4\rangle. \tag{6.430}$$

5.17 Descendants of 5.49

Let L be an immediate descendant of 5.49 order p^6. Then L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 is generated by p^2a modulo L_4, and L_4 has order p and is generated by p^3a. We also have baa, bab and pb in L_4. Subtracting a suitable scalar multiples of p^2a from b we may assume that $pb = 0$. If $bab \neq 0$ then we can assume that $bab = p^3a$ or ωp^3a and that $baa = 0$. And if $bab = 0$ then we can assume that $baa = 0$ or p^3a.

$$\langle a, b \mid baa,\ bab,\ pb,\ \text{class } 4\rangle, \tag{6.431}$$

$$\langle a, b \mid baa - p^3a,\ bab,\ pb,\ \text{class } 4\rangle, \tag{6.432}$$

$$\langle a, b \mid baa,\ bab - p^3a,\ pb,\ \text{class } 4\rangle, \tag{6.433}$$

$$\langle a, b \mid baa,\ bab - \omega p^3a,\ pb,\ \text{class } 4\rangle. \tag{6.434}$$

5.18 Descendants of 5.50

Let L be an immediate descendant of 5.50 order p^6. Then L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 is generated by baa modulo L_4, and L_4 has order p and is generated by $baaa$. We also have bab, p^2a and pb in L_4. If $p^2a \neq 0$ we can assume that $pb = 0$, and scaling b we can assume that $p^2a = baaa$. On the other hand, if $p^2a = 0$ then we have $pb = 0$ or $baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. Let $bab = \lambda baaa$.

If $p^2a = pb = 0$ then we can assume that $\lambda = 0$ or 1, giving

$$\langle a, b \mid bab,\ p^2a,\ pb,\ \text{class } 4\rangle, \tag{6.435}$$

$$\langle a, b \mid bab - baaa,\ p^2a,\ pb,\ \text{class } 4\rangle. \tag{6.436}$$

If $p^2a = baaa$ then we can assume that $\lambda = 0, 1, \omega$ or (in the case when $p = 1 \bmod 4$) ω^2 or ω^3. So we have

$$\langle a, b \mid bab,\ p^2a - baaa,\ pb,\ \text{class } 4\rangle, \tag{6.437}$$

$$\langle a, b \mid bab - baaa,\ p^2a - baaa,\ pb,\ \text{class } 4\rangle, \tag{6.438}$$

$$\langle a, b \mid bab - \omega baaa,\ p^2a - baaa,\ pb,\ \text{class } 4\rangle, \tag{6.439}$$

$$\langle a, b \mid bab - \omega^2 baaa,\ p^2a - baaa,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 4), \tag{6.440}$$

$$\langle a, b \mid bab - \omega^3 baaa,\ p^2a - baaa,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 4). \tag{6.441}$$

If $pb \neq 0$ but $p^2a = bab = 0$ we have

$$\langle a, b \mid bab,\ p^2a,\ pb - baaa,\ \text{class } 4\rangle, \tag{6.442}$$

$$\langle a, b \mid bab,\ p^2a,\ pb - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.443}$$

$$\langle a, b \mid bab,\ p^2a,\ pb - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3). \tag{6.444}$$

And ønally, if $p^2a = 0$ but bab and pb are non-zero, then we can take $bab = baaa$, and then we have

$$\langle a, b \mid bab - baaa,\ p^2a,\ pb - baaa,\ \text{class } 4\rangle, \tag{6.445}$$

$$\langle a, b \mid bab - baaa,\ p^2a,\ pb - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.446}$$

$$\langle a, b \mid bab - baaa,\ p^2a,\ pb - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3). \tag{6.447}$$

5.19 Descendants of 5.51

Let L be an immediate descendant of 5.51 order p^6. Then L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 is generated by baa modulo L_4, and L_4 has order p and is generated by $baaa$. We also have bab, p^2a and $pb - baa$ in L_4.

If we let $a' = a - \beta b$ then $ba'a' = baa - \beta bab$ and so if $bab \neq 0$ then we can assume that $pb = baa$. And if $p^2a \neq 0$ then subtracting a scalar multiple of pa from b we can similarly assume that $pb = baa$. But if $bab = p^2a = 0$ then we have $pb = baa + \lambda baaa$ where replacing a by $-a$ changes λ to $-\lambda$. So we have $(p+1)/2$ algebras

$$\langle a, b \mid bab,\ p^2a,\ pb - baa - \lambda baaa,\ \text{class } 4\rangle\ (0 \leq \lambda \leq (p-1)/2), \tag{6.448}$$

with λ and $-\lambda$ giving isomorphic algebras.

So suppose that $bab = \mu baaa$, and $p^2a = \nu baaa$, $pb = baa$, with at least one of μ, ν non-zero. We can assume that $\mu = 0$ or 1, and if $\mu = 0$ we can assume that $\nu = 1$. So we have

$$\langle a, b \mid bab,\ p^2a - baaa,\ pb - baa,\ \text{class } 4\rangle, \tag{6.449}$$

and p algebras

$$\langle a, b \mid bab - baaa,\ p^2a - \nu baaa,\ pb - baa,\ \text{class } 4\rangle\ (0 \leq \nu < p). \tag{6.450}$$

5.20 Descendants of 5.52

Let L be an immediate descendant of 5.52 order p^6. This case is almost identical to the previous case. As above, L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 is generated by baa modulo L_4, and L_4 has order p and is generated by $baaa$. We also have bab, p^2a and $pb - \omega baa$ in L_4.

If we let $a' = a - \beta b$ then $ba'a' = baa - \beta bab$ and so if $bab \neq 0$ then we can assume that $pb = \omega baa$. And if $p^2a \neq 0$ then subtracting a scalar multiple of pa from b we can similarly assume that $pb = \omega baa$. But if $bab = p^2a = 0$ then we have $pb = \omega baa + \lambda baaa$ where replacing a by $-a$ changes λ to $-\lambda$. So we have $(p+1)/2$ algebras

$$\langle a, b \mid bab,\ p^2a,\ pb - \omega baa - \lambda baaa,\ \text{class } 4\rangle\ (0 \leq \lambda \leq (p-1)/2), \tag{6.451}$$

with λ and $-\lambda$ giving isomorphic algebras.

So suppose that $bab = \mu baaa$, and $p^2a = \nu baaa$, $pb = baa$, with at least one of μ, ν non-zero. We can assume that $\mu = 0$ or 1, and if $\mu = 0$ we can assume that $\nu = 1$. So we have

$$\langle a, b \mid bab,\ p^2a - baaa,\ pb - \omega baa,\ \text{class } 4\rangle, \tag{6.452}$$

and p algebras

$$\langle a, b \mid bab - baaa,\ p^2a - \nu baaa,\ pb - \omega baa,\ \text{class } 4\rangle\ (0 \leq \nu < p). \tag{6.453}$$

5.21 Descendants of 5.53

This algebra is terminal.

5.22 Descendants of 5.54

Let L be an immediate descendant of 5.54 order p^6. Then L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 is generated by bab modulo L_4, and L_4 has order p and is generated by $babb$. We also have baa, p^2a and pb in L_4.

If $p^2a \neq 0$ then we can assume that $pb = 0$. We can assume that $baa = 0$ or $babb$.

Consider the case when $baa = 0$. If $p^2a \neq 0$ then we can assume that $p^2a = babb$ or (in the case when $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. And if $p^2a = 0$ then we can assume that $pb = 0$ or $babb$.

Now consider the case when $baa = babb$. If $p^2a \neq 0$ then $p^2a = babb$ or (in the case when $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. And if $p^2a = 0$ then we can assume that $pb = 0$, $babb$, $\omega babb$ or (in the case when $p = 1 \bmod 4$) $\omega^2 babb$ or $\omega^3 babb$. So we have

$$\langle a, b \mid baa,\ p^2a,\ pb,\ \text{class } 4\rangle, \tag{6.454}$$

$$\langle a, b \mid baa,\ p^2a,\ pb - babb,\ \text{class } 4\rangle, \tag{6.455}$$

$$\langle a, b \mid baa,\ p^2a - babb,\ pb,\ \text{class } 4\rangle, \tag{6.456}$$

$$\langle a, b \mid baa,\ p^2a - \omega babb,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.457}$$

$$\langle a, b \mid baa,\ p^2a - \omega^2 babb,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.458}$$

$$\langle a, b \mid baa - babb,\ p^2a,\ pb,\ \text{class } 4\rangle, \tag{6.459}$$

$$\langle a, b \mid baa - babb,\ p^2a,\ pb - babb,\ \text{class } 4\rangle, \tag{6.460}$$

$$\langle a, b \mid baa - babb,\ p^2a,\ pb - \omega babb,\ \text{class } 4\rangle, \tag{6.461}$$

$$\langle a, b \mid baa - babb,\ p^2a,\ pb - \omega^2 babb,\ \text{class } 4\rangle\ (p = 1 \bmod 4), \tag{6.462}$$

$$\langle a, b \mid baa - babb,\ p^2a,\ pb - \omega^3 babb,\ \text{class } 4\rangle\ (p = 1 \bmod 4), \tag{6.463}$$

$$\langle a, b \mid baa - babb,\ p^2a - babb,\ pb,\ \text{class } 4\rangle, \tag{6.464}$$

$$\langle a, b \mid baa - babb,\ p^2a - \omega babb,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \tag{6.465}$$

$$\langle a, b \mid baa - babb,\ p^2a - \omega^2 babb,\ pb,\ \text{class } 4\rangle\ (p = 1 \bmod 3). \tag{6.466}$$

5.23 Descendants of 5.55 ˘ 5.57

These three algebras are terminal.

5.24 Descendants of 5.58

Let L be an immediate descendant of 5.58 of order p^6. Then L is generated by a, b, L_2 is generated by ba, pa modulo L_3, and L_3 is generated by p^2a modulo L_4, and L_4 has order p and is generated by p^3a. We also have baa and $pb - ba$ in L_4.

Subtracting a suitable scalar multiple of p^2a from b we may assume that $pb = ba$. And we may assume that $baa = 0$ or p^3a. So we have

$$\langle a, b \mid baa,\ pb - ba,\ \text{class } 4\rangle, \tag{6.467}$$

$$\langle a, b \mid baa - p^3a,\ pb - ba,\ \text{class } 4\rangle. \tag{6.468}$$

5.25 Descendants of 5.59

This algebra is terminal.

5.26 Descendants of 5.60

Let L be an immediate descendant of 5.60 of order p^6. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5, and L_5 has order p and is generated by $baaaa$, $baaab$. We also have bab, pa and pb in L_5.

We can assume that the commutator structure is the same as one of the algebras 6.30 ˘ 6.32 from the list of nilpotent Lie algebras over $\mathbb{Z}_p$ of dimension 6. So we can assume that one of the following sets of commutator relations holds.

$$\begin{aligned} bab &= baaab = 0, \\ bab &= baaaa,\ baaab = 0, \\ bab &= baaaa = 0. \end{aligned}$$

5.26.1 Case 1

Suppose that $bab = baaab = 0$. Then L_5 is generated by $baaaa$ and $pa = \lambda baaaa$, $pb = \mu baaaa$ for some λ, μ. If $pb \neq 0$ then we can assume that $pa = 0$ and we can assume that $pb = baaaa$, $\omega baaaa$ or (in the case when $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. And if $pb = 0$ then we can assume that $pa = 0$ or $baaaa$. So we have

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb,\ \text{class } 5\rangle, \tag{6.469}$$

$$\langle a, b \mid bab,\ baaab,\ pa - baaaa,\ pb,\ \text{class } 5\rangle, \tag{6.470}$$

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb - baaaa,\ \text{class } 5\rangle, \tag{6.471}$$

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb - \omega baaaa,\ \text{class } 5\rangle, \tag{6.472}$$

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4), \tag{6.473}$$

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb - \omega^3 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4). \tag{6.474}$$

5.26.2 Case 2

Suppose that $bab = baaaa$, $baaab = 0$. As above, L_5 is generated by $baaaa$ and $pa = \lambda baaaa$, $pb = \mu baaaa$ for some λ, μ. If $pb \neq 0$ then we can assume that $pa = 0$ and we can assume that $pb = baaaa$, $\omega baaaa$ or (in the case when $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. And if $pb = 0$ then we can assume that $pa = 0$ or $baaaa$ or $\omega baaaa$ or (when $p = 1 \bmod 6$) $\omega^2 baaaa$ or $\omega^3 baaaa$ or $\omega^4 baaaa$ or $\omega^5 baaaa$ or . So we have

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb,\ \text{class } 5\rangle, \tag{6.475}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - baaaa,\ pb,\ \text{class } 5\rangle, \tag{6.476}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - \omega baaaa,\ pb,\ \text{class } 5\rangle, \tag{6.477}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - \omega^2 baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3), \tag{6.478}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - \omega^3 baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3), \tag{6.479}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - \omega^4 baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3), \tag{6.480}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - \omega^5 baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3), \tag{6.481}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb - baaaa,\ \text{class } 5\rangle, \tag{6.482}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb - \omega baaaa,\ \text{class } 5\rangle, \tag{6.483}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4), \tag{6.484}$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb - \omega^3 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4). \tag{6.485}$$

5.26.3 Case 3

Finally, suppose that $bab = baaaa = 0$. So L_5 is generated by $baaab$ and $pa = \lambda baaab$, $pb = \mu baaab$ for some λ, μ. If $pb \neq 0$ then we may assume that $pb = baaab$ and that $pa = 0$, $baaab$, $\omega baaab$ or (in the case when $p = 1 \bmod 4$) $\omega^2 baaab$ or $\omega^3 baaab$. And if $pb = 0$ then we can assume that $pa = 0$, $baaab$, $\omega baaab$. So we have

$$\langle a, b \mid bab,\ baaaa,\ pa,\ pb,\ \text{class } 5\rangle, \tag{6.486}$$

$$\langle a, b \mid bab,\ baaaa,\ pa - baaab,\ pb,\ \text{class } 5\rangle, \tag{6.487}$$

$$\langle a, b \mid bab,\ baaaa,\ pa - \omega baaab,\ pb,\ \text{class } 5\rangle, \tag{6.488}$$

$$\langle a, b \mid bab,\ baaaa,\ pa,\ pb - baaab,\ \text{class } 5\rangle, \tag{6.489}$$

$$\langle a, b \mid bab,\ baaaa,\ pa - baaab,\ pb - baaab,\ \text{class } 5\rangle, \tag{6.490}$$

$$\langle a, b \mid bab,\ baaaa,\ pa - \omega baaab,\ pb - baaab,\ \text{class } 5\rangle, \tag{6.491}$$

$$\langle a, b \mid bab,\ baaaa,\ pa - \omega^2 baaab,\ pb - baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 4), \tag{6.492}$$

$$\langle a, b \mid bab,\ baaaa,\ pa - \omega^3 baaab,\ pb - baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 4). \tag{6.493}$$

5.27 Descendants of 5.61 ˘ 564

These algebras are all terminal.

5.28 Descendants of 5.65

Let L be an immediate descendant of 5.65 of order p^6. Then L is generated by a, b, L_2 is generated by ba modulo L_3, and L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5, and L_5 has order p and is generated by $baaaa$, $baaab$. We also have $bab - baaa$, pa and pb in L_5.

The commutator structure of L must be the same as one of the algebras 6.33 or 6.44 from the list of nilpotent Lie algebras over $\mathbb{Z}_p$ of dimension 6. So we may assume that $bab = baaa$, and that either $baaaa = 0$ or $baaab = 0$.

5.28.1 $baaaa = 0$

Let $bab = baaa$ and let $baaaa = 0$. Then L_5 is generated by $baaab$ and $pa = \lambda baaab$, $pb = \mu baaab$ for some λ, μ. If a', b' generate L and a', b' satisfy the same commutator relations as a, b then $a' = \alpha a$ modulo L_2 and $b' = \alpha^2 b$ modulo L_2 for some $\alpha \neq 0$. Then

$$\begin{aligned} pa' &= \alpha pa = \alpha^{-6}\lambda b'a'a'a'b', \\ pb' &= \alpha^2 pb = \alpha^{-5}\mu b'a'a'a'b'. \end{aligned}$$

One possibility is that $\lambda = \mu = 0$:

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa,\ pb,\ \text{class } 5\rangle. \tag{6.494}$$

If $\lambda = 0$ and $\mu \neq 0$ then we have

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa,\ pb - baaab,\ \text{class } 5\rangle, \tag{6.495}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa,\ pb - \omega baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \tag{6.496}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa,\ pb - \omega^2 baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \tag{6.497}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa,\ pb - \omega^3 baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \tag{6.498}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa,\ pb - \omega^4 baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 5). \tag{6.499}$$

And if $\lambda \neq 0$ and $\mu = 0$ then we have

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - baaab,\ pb,\ \text{class } 5\rangle, \tag{6.500}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - \omega baaab,\ pb,\ \text{class } 5\rangle, \tag{6.501}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - \omega^2 baaab,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3), \tag{6.502}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - \omega^3 baaab,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3), \tag{6.503}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - \omega^4 baaab,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3), \tag{6.504}$$

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - \omega^5 baaab,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3). \tag{6.505}$$

Finally consider the case when λ and μ are both non-zero. If $p \neq 1$ modulo 5, then we can take $\mu = 1$ and then we have $p-1$ algebras

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - \lambda baaab,\ pb - baaab,\ \text{class } 5\rangle\ (0 < \lambda < p,\ p \neq 1 \bmod 5). \tag{6.506}$$

And if $p = 1$ modulo 5, then we can take $\mu = 1, \omega, \omega^2, \omega^3$, or ω^4 and for any give value of μ, dicerent values λ, λ' give isomorphic algebras if $\lambda^5 = \lambda'^5$. So we again have $p-1$ algebras

$$\langle a, b \mid bab - baaa,\ baaaa,\ pa - \lambda baaab,\ pb - \mu baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \tag{6.506A}$$

where $\mu = 1, \omega, \omega^2, \omega^3$, or ω^4 and $0 < \lambda < p$, with λ running over a set of representatives for the equivalence classes for the equivalence relation

$$\lambda \sim \lambda' \text{ if } \lambda^5 = \lambda'^5.$$

5.28.2 $baaab = 0$

Let $bab = baaa$ and let $baaab = 0$. Then L_5 is generated by $baaaa$ and $pa = \lambda baaaa$, $pb = \mu baaaa$ for some λ, μ. If a', b' generate L and a', b' satisfy the same commutator relations as a, b then $a' = \alpha a$ modulo L_2 and $b' = \alpha^2 b$ modulo L_2 for some $\alpha \neq 0$. Then

$$\begin{aligned} pa' &= \alpha pa = \alpha^{-5}\lambda b'a'a'a'a', \\ pb' &= \alpha^2 pb = \alpha^{-4}\mu b'a'a'a'a'. \end{aligned}$$

So we obtain

$$\langle a, b \mid bab - baaa,\ baaab,\ pa,\ pb,\ \text{class } 5\rangle, \tag{6.507}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa,\ pb - baaaa,\ \text{class } 5\rangle, \tag{6.508}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa,\ pb - \omega baaaa,\ \text{class } 5\rangle, \tag{6.509}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa,\ pb - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4), \tag{6.510}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa,\ pb - \omega^3 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4), \tag{6.511}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa - baaaa,\ pb,\ \text{class } 5\rangle, \tag{6.512}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa - \omega baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \tag{6.513}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa - \omega^2 baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \tag{6.514}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa - \omega^3 baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \tag{6.515}$$
$$\langle a, b \mid bab - baaa,\ baaab,\ pa - \omega^4 baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 5). \tag{6.516}$$

If $p \neq 1$ modulo 5, then we can take $\lambda = 1$ and then we have $p-1$ algebras

$$\langle a, b \mid bab - baaa,\ baaab,\ pa - baaaa,\ pb - \mu baaaa,\ \text{class } 5\rangle\ (0 < \mu < p,\ p \neq 1 \bmod 5). \tag{6.517}$$

And if $p = 1$ modulo 5, then we can take $\lambda = 1, \omega, \omega^2, \omega^3$, or ω^4 and for any give value of λ, diœerent values μ, μ' give isomorphic algebras if $\mu^5 = \mu'^5$. So we again have $p - 1$ algebras

$$\langle a, b \mid bab - baaa,\ baaab,\ pa - \lambda baaaa,\ pb - \mu baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 5), \quad (6.517A)$$

where $\lambda = 1, \omega, \omega^2, \omega^3$, or ω^4 and $0 < \mu < p$, with μ running over a set of representatives for the equivalence classes for the equivalence relation

$$\mu \sim \mu' \text{ if } \mu^5 = \mu'^5.$$

5.29 Descendants of 5.66 ˘ 5.72

These algebras are all terminal.

5.30 Descendants of 5.73

Let L be an immediate descendant of 5.73 of order p^6. Then L is generated by a, b, L_2 is generated by pa modulo L_3, and L_3 is generated by p^2a modulo L_4, L_4 is generated by p^3a modulo L_5, and L_5 has order p and is generated by p^4a. We also have ba and pb in L_5.

Subtracting a suitable scalar multiple of p^3a from b we may assume that $pb = 0$. And we may assume that $ba = 0$ or p^4a. So we have

$$\langle a, b \mid ba,\ pb,\ \text{class } 5\rangle, \quad (6.518)$$

$$\langle a, b \mid ba - p^4a,\ pb,\ \text{class } 5\rangle, \quad (6.519)$$

5.31 Descendants of 5.74

This algebra is terminal.