On the uniqueness of the generalized octagon of order (2,4)

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Dedicated to the memory of Ákos Seress

ABSTRACT. The smallest known thick generalized octagon has order (2,4) and can be constructed from the parabolic subgroups of the Ree group ${}^2\mathrm{F}_4(2)$. It is not known whether this generalized octagon is unique up to isomorphism. We show that it is unique up to isomorphism among those having a point a whose stabilizer in the automorphism group both fixes setwise every line on a and contains a subgroup that is regular on the set of 1024 points at maximal distance to a. Our proof uses extensively the classification of the groups of order dividing 2^9.

1. Introduction

Recall that a point-line geometry $\mathcal{G} = (P, \mathcal{L})$ consists of a set P of *points*, and a collection $\mathcal{L}$ of subsets of P, each of size at least two, called *lines*. A point $p \in P$ is *incident* with a line $L \in \mathcal{L}$ if p is an element of L. Two points incident to the same line are *collinear*. We associate two graphs with every point-line geometry. The *incidence graph* of $\mathcal{G}$ has as vertices all points and all lines of $\mathcal{G}$, with edges connecting incident point-line pairs. The *collinearity graph* of $\mathcal{G}$ has as vertices all points of $\mathcal{G}$, with edges connecting collinear pairs of points. It is easy to see that the incidence graph is connected if and only if the collinearity graph is connected; if so, then $\mathcal{G}$ is *connected*.

The *diameter* of a connected graph Γ is the maximal distance between vertices and its *girth* is the shortest length of a cycle.

Definition 1.1. For $n \geq 3$, a *generalized* n*-gon* is a point-line geometry $\mathcal{G}$ satisfying the following properties:

(1) the diameter of the incidence graph Γ of $\mathcal{G}$ is n;
(2) the girth of Γ is $2n$;
(3) $\mathcal{G}$ is regular: every point is incident with the same number $t + 1 > 1$ of lines and every line is incident with the same number $s + 1 > 1$ of points.

This concept was introduced and developed by Tits [13, 14].

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We are interested in finite generalized n-gons when both s and t are finite. The pair (s,t) is the *order* of $\mathcal{G}$. The generalized n-gon $\mathcal{G}$ is *thin* if $s=1=t$. For every n there exists exactly one thin generalized n-gon, which can be described as the geometry of all vertices and edges of the usual n-gon.

By a famous theorem of Feit and Higman [10], a finite generalized n-gon with $n \geq 3$ is either thin, or satisfies $n \in \{3,4,6,8,12\}$. Furthermore, if $\mathcal{G}$ is *thick* (that is, both $s,t \geq 2$), then $n=12$ is impossible. Thus the *gonality* n of a thick finite generalized n-gon is at most eight. This largest gonality, $n=8$, is the only case where the smallest thick generalized n-gons are not known up to isomorphism. The smallest thick finite generalized quadrangles and hexagons are unique (see [7] and [12]).

It follows from [10] that the smallest order for which a thick finite generalized octagon can exist is $(2,4)$. A generalized octagon of order $(2,4)$ was constructed by Tits [15] as part of an infinite series of generalized octagons related to the groups ${}^2\mathrm{F}_4(q)$. The octagon of order $(2,4)$ is obtained from ${}^2\mathrm{F}_4(2)$ by taking for points the maximal parabolic subgroups G_1 for which $G_1/O_2(G_1)$ is a Frobenius group of order 20, and by taking for lines the maximal parabolics G_2 of the other type, with incidence between G_1 and G_2 defined by $O_2(G_1) \subseteq G_2$; more details are given in Example 1.3.

The uniqueness, or otherwise, of the generalized octagon of order $(2,4)$ remains an open and very difficult problem. In [8], De Bruyn proved that the example of Tits is the only one in which the unique generalized octagon of order $(2,1)$ embeds. In [3], it is shown that it is the only one of order $(2,4)$ admitting a vertex-transitive group of automorphisms.

In this paper, we establish its uniqueness under another assumption. Suppose $\mathcal{G} = (P,\mathcal{L})$ is a generalized octagon. Since the incidence graph Γ is by definition bipartite and the gonality of $\mathcal{G}$ is even, if two vertices of Γ are at the maximal distance, eight, then these have the same type: both are points or lines. Elements of $\mathcal{G}$ at maximal distance in Γ are *opposite* (cf. [17, p. 5]).

Theorem 1.2. *Let $\mathcal{G} = (P,\mathcal{L})$ be a generalized octagon of order $(2,4)$. Assume that it admits a group Q of automorphisms of $\mathcal{G}$ such that, for some $a \in P$,*

(i) Q fixes a and stabilizes setwise every line on a;
(ii) Q is transitive on the set of points opposite to a.

Then $\mathcal{G}$ is the generalized octagon related to the Ree group ${}^2\mathrm{F}_4(2)$.

Example 1.3. In Table 1, we summarise the defining relations, given in [15, p. 326], for ${}^2\mathrm{F}_4(2)$ on generators $u_1,\dots,u_8$, $v_1,\dots,v_8$, r_1, and r_8.

We denote this group by T. Let Q be its subgroup generated by $u_2,\dots,u_8$, let U be its subgroup generated by Q and u_1, let G_1 be its subgroup generated by U and r_1, and let G_2 be its subgroup generated by U and r_8. Then $Q = O_2(G_1)$, and the elements $g = u_1^{-1}v_1$ and $f = v_1$ generate a Frobenius group of order 20 that is a complement to Q in G_1.

Since G_1 and G_2 are representatives of the two types of maximal parabolic subgroups of T and both contain Q, the construction mentioned above gives the known octagon of order $(2,4)$. Since G_1 normalises Q, the latter fixes a unique point a of the generalized octagon. It is easily verified that Q satisfies the conditions of Theorem 1.2. Observe that

$u_1^4 = u_2^2 = u_3^4 = u_4^2 = 1$	$u_5^4 = u_6^2 = u_7^4 = u_8^2 = 1$	$v_1^4 = v_2^2 = v_3^4 = v_4^2 = 1$
$v_5^4 = v_6^2 = v_7^4 = v_8^2 = 1$	$[u_1, u_2] = [u_1, u_5] = 1$	$[u_2, u_4] = [u_2, u_6] = 1$
$u_3u_1 = u_1u_2u_3$	$u_4u_1 = u_1u_3^2u_4$	$u_8u_2 = u_2u_4u_6u_8$
$u_6u_1 = u_1u_3^2u_4u_5^2u_6$	$u_7u_1 = u_1u_2u_3^3u_5u_7$	$u_8u_1 = u_1u_2u_3^2u_4u_5^3u_6u_7u_8$
$r_1 = u_1v_1^2u_1^{-1}$	$r_8 = u_8v_8u_8^{-1}$	$(r_1r_8)^8 = 1$
$r_1u_1r_1 = v_1$	$r_1u_2r_1 = u_8$	$r_1u_3r_1 = u_7$
$r_1u_4r_1 = u_6$	$r_1u_5r_1 = u_5$	$r_1v_2r_1 = v_8$
$r_1v_3r_1 = v_7$	$r_1v_4r_1 = v_6$	$r_1v_5r_1 = v_5$
$r_8u_1r_8 = u_7$	$r_8u_2r_8 = u_6$	$r_8u_3r_8 = u_5$
$r_8u_4r_8 = u_4$	$r_8u_8r_8 = v_8$	$r_8v_1r_8 = v_7$
$r_8v_2r_8 = v_6$	$r_8v_3r_8 = v_5$	$r_8v_4r_8 = v_4$

TABLE 1. Defining relations for ${}^2\mathrm{F}_4(2)$

g cyclically permutes the five lines on a and f fixes a unique line on a and cyclically permutes the other four.

A motivation for the theorem was Kantor's construction [11] of generalized 4-gons using a group similar to Q; we discuss this in Example 3.4.

2. Preliminaries

All graphs in this paper are undirected, and their edges have distinct vertices. Throughout, $\mathcal{G} = (P, \mathcal{L})$ is a generalized octagon of order $(2, 4)$, with incidence graph Γ and collinearity graph Δ. We mostly work with Δ, hence our first task is to restate what is known about $\mathcal{G}$ in terms of Δ. A k*-clique* of a graph is a set of k pairwise adjacent vertices. Every line in $\mathcal{L}$ induces a 3-clique in Δ. A *path of length* k from a to b is a sequence $a = a_0, a_1, \ldots, a_k = b$ of vertices of Γ such that $\{a_{i-1}, a_i\}$ is an edge for $i = 1, \ldots, k$. The path is *simple* whenever $a_{i-1} \neq a_{i+1}$ for $i = 1, \ldots, k-1$. The *distance* between two subsets of points of Δ is the minimum among all distances between a point from one and a point from the second.

The first condition defining a generalized n-gon implies that the diameter of Δ is $n/2 = 4$. In particular, points are opposite if and only if they are at distance 4 in Δ. The second condition implies that the geometric girth of Δ is $n = 8$; the *geometric girth* is the shortest length of a cycle satisfying the extra condition that no three consecutive vertices lie in the same clique.

Lemma 2.1. *For each point p and line L, there is a unique point in L closest to p in Γ. Dually, there exists a unique line containing p closest to L in Γ. Moreover, if two points are connected by a simple path of length four in Δ, then they are opposite.*

Proof: Since Γ is bipartite and the gonality 8 is even, p and L cannot be opposite. In particular, in Γ they are connected by a unique shortest path, since the girth of Γ is 16. The unique point of L closest to p is the neighbour of L on this path and, symmetrically, the unique line on p closest to L is the neighbour of p.

If the two points were connected by a path of length at most three in Δ, then this path, together with the given path, would make a cycle of geometric length at most 7 in

Δ, a contradiction. The final statement follows. □

Corollary 2.2. *Every two points, p and q, that are opposite are connected by exactly $5 = t + 1$ shortest paths in Δ, one through each line on p and, symmetrically, one through each line on q.*

A dual statement also holds for lines that are opposite. We show that the Δ-distance between opposite lines is three.

Lemma 2.3. *Suppose L and M are lines that are opposite. For each $p \in L$, the unique $q \in M$ closest to p is at distance three from p in Δ. The map $L \to M$ sending every point of L to its closest point on M is a bijection from L onto M. In particular, L and M are connected by exactly $3 = s + 1$ shortest paths, one through each point of L and, symmetrically, one through each point of M.*

Proof: Let $p \in L$. By Lemma 2.1, M contains a unique point q closest to p. Clearly, the distance $d(p, q)$ in Δ between p and q is less than the diameter, hence $d(p, q) \leq 3$. If $d(p, q) \leq 2$, then in Γ the points p and q are at distance at most four, which implies that L and M are at distance at most six; a contradiction since L and M are opposite. Therefore $d(p, q) = 3$.

Symmetrically, L contains a unique point closest to q, so clearly that point is p. This establishes the bijection. The last claim is an easy consequence. □

3. Points opposite to a given point

Fix $a \in P$. Let $P_a = \{b \in P \mid d(a, b) = 4\}$, the set of points opposite to a. In this section we study the subgraph of Δ induced on this set.

Lemma 3.1. $|P_a| = 2^{10}$.

Proof: This follows from a standard argument for finite generalized n-gons, which can be found, for instance, in [17, Lemma 1.5.4]. □

Let Σ be the subgraph induced on P_a. We showed in the above proof that every line on $b \in P_a$ has exactly one point in $\Delta_3(a)$.

Lemma 3.2. *The graph Σ has valency five. It contains no 3-clique and has girth at least eight.*

Proof: Only the last claim requires comment. Since Σ contains no 3-clique, every cycle in Σ satisfies the condition that no three consecutive vertices on it lie on the same line of $\mathcal{G}$. The geometric girth of Δ is eight, so the claim follows. □

Corollary 3.3. *Each connected component of Σ has at least 170 vertices.*

Proof: Picking a vertex b of Σ and setting $\Sigma_k(b)$ to be the set of vertices of Σ that are at distance k from b (in Σ, which is not necessarily the same distance in Δ), we learn that $|\Sigma_1(b)| = 5$, $|\Sigma_2(b)| = |\Sigma_1(b)| \cdot 4 = 20$, $|\Sigma_3(b)| = |\Sigma_2(b)| \cdot 4 = 80$, and finally,

$|\Sigma_4(b)| \geq |\Sigma_3(b)| \cdot 4/5 = 64$. The last inequality uses that every vertex from $\Sigma_4(b)$ is adjacent to at most five vertices in $\Sigma_3(b)$. Thus the connected component of Σ containing b has at least $1 + 5 + 20 + 80 + 64 = 170$ vertices. □

Example 3.4. We digress to show how the Tits generalized octagon of order $(2, 4)$ can be obtained directly from the group Q of Example 1.3, by employing Kantor's generalized 4-gon construction. Consider the following subgroups of Q:

$$Q_j^{(1)} = \langle u_8, u_7, \ldots, u_{9-j} \rangle \qquad \text{for } 1 \leq j \leq 6,$$

and $Q_j^{(i)}$ for $2 \leq i \leq 5$ defined by $Q_j^{(k+1)} = \left(Q_j^{(k)}\right)^g$, where $k \in \{1, 2, 3, 4\}$ and g is as in Example 1.3. These subgroups satisfy the following properties with $n = 8$, $s = 2$, and $t = 4$, where $Q_0^{(i)} = 1$ and $Q_{n-1}^{(i)} = Q$.

(O) For $1 \leq i \leq t+1$ and $1 \leq j \leq n-1$, $Q_{j-1}^{(i)}$ is a subgroup of $Q_j^{(i)}$ of index t or s depending on whether $n - j$ is even or odd.

(C) For indices $i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, \ldots, 5\}$ such that $i_m \neq i_{m+1}$ for $1 \leq m \leq k-1$ and $i_1 \neq i_k$ and $j_1 + \cdots + j_k = n - 1$,

$$1 \notin \left(Q_{j_1}^{(i_1)}\right)^{\#} \cdots \left(Q_{j_k}^{(i_k)}\right)^{\#}$$

where $G^{\#} := G \setminus \{1\}$. With this data $\mathcal{Q} = \left(Q_j^{(i)}\right)_{i,j}$, we construct a graph $\Gamma(\mathcal{Q})$ as follows. Its vertices are the $t + 2$ labels a and L_i for $i = 1, \ldots, t+1$, and all right cosets of $Q_j^{(i)}$ in Q for $1 \leq i \leq t+1$ and $0 \leq j \leq n-2$. Its edges are the pairs $\{a, L_i\}$; the pairs $\{L_i, Q_{n-2}^{(i)}x\}$; and all pairs $\{Q_j^{(i)}x, Q_{j+1}^{(i)}y\}$ with $Q_j^{(i)}x \subset Q_{j+1}^{(i)}y$ for $1 \leq i \leq t+1$, $x, y \in Q$, and $0 \leq j \leq n-3$.

Now every graph $\Gamma(\mathcal{Q})$ for which $\mathcal{Q}$ satisfies (O) and (C) is the incidence graph of a generalized n-gon of order (s, t). In particular, the above collection $\mathcal{Q}$ of subgroups of Q gives another construction of the known generalized 8-gon.

For $n = 3$, Conditions (O) and (C) are easily seen to be equivalent to the existence of a translation plane structure on Q. For $n = 4$, the conditions translate to those formulated by Kantor in [11]. For $n = 6$, they have been used in [6].

4. Edge colours

Let Σ be the graph induced by Δ on the set of points P_a opposite to a fixed $a \in P$. We arbitrarily attach colours 1 to 5 to the five lines on a. Consider an edge $\{b, c\}$ in Σ and let $L \in \mathcal{L}$ be the line containing b and c. Lemma 2.1 shows that there is a unique line on a closest to L. We colour $\{b, c\}$ with the colour of that line. Thus every edge in Σ is given a colour from 1 to 5. Corollary 2.2 implies that the five edges on every vertex of Σ exhibit all five colours. Hence no two edges incident to the same vertex can have the same colour.

Let $\mathcal{G}$ admit an automorphism group Q that fixes a, stabilizes every line on a, and is transitive on the set P_a of opposites to a. Clearly, Q acts on Σ. Since it stabilizes each of the five lines on a, it permutes the edges of Σ preserving each colour.

Lemma 4.1. *The group Q acts regularly on P_a.*

Proof: In view of assumption (ii) of Theorem 1.2, we need only show that Q_b is trivial for $b \in P_a$. Since the five edges of Σ on b have pairwise different colours, Q_b fixes all neighbours of b in Σ: that is, $Q_b = Q_c$ whenever b and c are adjacent vertices of Σ. Thus Q_b fixes pointwise the connected component Σ_0 of Σ containing b.

Since Q_b fixes the five edges on b and since lines in Γ have three points, Q_b fixes all points of Γ collinear with b, and the same applies to all vertices of Σ_0. Picking an arbitrary n-path $x_0, x_1, \dots, x_n$ in Σ, we see that Q_b fixes every point on this path and furthermore fixes every neighbour of x_0 and of x_1. Thus, the assumptions of [16, (3.7)] are satisfied, and so Q_b is trivial. □

This, together with Lemma 3.1, determines the order of Q.

Corollary 4.2. *The group Q has order 2^{10}.*

We next show that Σ (the subgraph induced on P_a) is a Cayley graph for Q. We fix $b \in P_a$ as our initial vertex and let $\{b, c_i\}$ for $1 \leq i \leq 5$ be the five edges on b where $\{b, c_i\}$ has the colour i. Let α_i be the unique element of Q taking b to c_i.

Lemma 4.3. *Each α_i is an involution.*

Proof: The action of Q is colour-preserving and $\{b, c_i\}$ is the only edge on c_i of colour i; hence α_i stabilizes the edge $\{b, c_i\}$, and so takes c_i back to b. In particular, $b^{\alpha_i^2} = c_i^{\alpha_i} = b$. Hence $\alpha_i^2 \in Q_b = 1$. □

Let $\mathrm{Cay}(G, I)$ be the *Cayley digraph* of a group G with subset I. The *vertices* of this digraph are the elements of G and (x, y) is a *directed edge* whenever $yx^{-1} \in I$. Clearly, $\mathrm{Cay}(G, I)$ is an undirected graph if every element of I is an involution. In particular, in view of Lemma 4.3, $\mathrm{Cay}(Q, \{\alpha_1, \dots, \alpha_5\})$ is a graph.

Lemma 4.4. *The coloured graph Σ is isomorphic to $\mathrm{Cay}(Q, \{\alpha_1, \dots, \alpha_5\})$. The map ϕ assigning to each $c \in P_a$ the unique element of Q taking b to c is an isomorphism.*

Proof: Clearly, ϕ is a bijection between the vertex sets P_a and Q. Since both graphs have valency five, it remains to show that ϕ takes edges to edges. Consider an edge $\{d, e\}$ in Σ. Let $\beta = \phi(d)$, that is, $b^\beta = d$. If i is the colour of $\{d, e\}$, then β takes the edge $\{b, c_i\}$ to $\{d, e\}$, which means that $c_i^\beta = e$. But $c_i = b^{\alpha_i}$, so $e = b^{\alpha_i \beta}$, that is, $\phi(e) = \alpha_i \beta$. Thus $\phi(e)\phi(d)^{-1} = \alpha_i \beta \beta^{-1} = \alpha_i \in \{\alpha_1, \dots, \alpha_5\}$, which proves that $\phi(d)$ and $\phi(e)$ are adjacent in $\mathrm{Cay}(Q, \{\alpha_1, \dots, \alpha_5\})$. □

Lemmas 3.2 and 4.4 together imply the following.

Corollary 4.5. *The Cayley graph $\mathrm{Cay}(Q, \{\alpha_1, \dots, \alpha_5\})$ has girth at least eight. In particular, each product of at most seven α_i in which any two consecutive elements are distinct represents a non-identity element of Q.*

Let $Q_0 = \langle \alpha_1, \dots, \alpha_5 \rangle$. It is well known that $\mathrm{Cay}(G, I)$ is connected if and only if $G = \langle I \rangle$. Thus each connected component of $\mathrm{Cay}(Q, \{\alpha_1, \dots, \alpha_5\})$ is isomorphic to $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$. In particular, the size of each connected component of Σ is $|Q_0|$, and the number of connected components of Σ is the index $[Q : Q_0]$. Since Q is a 2-group, so is Q_0. Corollary 3.3 implies the following.

Corollary 4.6. *The order of Q_0 is 2^8, 2^9, or 2^{10}. The graph Σ has 4, 2, or 1 connected components, respectively.*

Example 4.7. We revisit Example 1.3, defining the following elements of T:

$$\alpha_0 = u_5 \qquad \alpha_1 = u_2 \qquad \alpha_2 = u_2u_3^3u_4u_5u_6u_7^2u_8$$
$$\alpha_3 = u_2u_3u_4u_5^2u_6u_7^3u_8 \qquad \alpha_4 = u_2u_3^2u_4u_5^3u_6u_7u_8 \qquad \alpha_5 = u_8$$

That $\alpha_0, \ldots, \alpha_5$ generate Q is a consequence of the following:

$$u_2 = \alpha_1 \qquad u_3 = \alpha_0\alpha_5\alpha_3\alpha_5\alpha_4 \qquad u_4 = \alpha_3\alpha_1\alpha_4\alpha_5\alpha_2 \qquad u_5 = \alpha_0$$
$$u_6 = \alpha_2\alpha_5\alpha_3\alpha_4\alpha_1 \qquad u_7 = \alpha_0\alpha_4\alpha_2\alpha_4\alpha_3 \qquad u_8 = \alpha_5$$

Let b be the image of G_1 under v_5; as the notation suggests, $\alpha_1, \ldots, \alpha_5$ are the elements of Q moving b to a collinear point in Σ. The Cayley graph Σ has two connected components, which are interchanged by α_0. This implies that Q_0 has order 2^9.

Here is a defining set of relations for Q in terms of these generators. It enables us to study Q and Σ without recourse to T. We write $[x, y] = x^{-1}y^{-1}xy$ for $x, y \in Q$.

$$\begin{aligned}
\alpha_1^2 &= \alpha_2^2 = \cdots = \alpha_5^2 = 1, \\
\alpha_0^2 &= \alpha_5\alpha_2\alpha_4\alpha_1\alpha_3, \\
\alpha_1^{\alpha_0} &= \alpha_3\alpha_1\alpha_4\alpha_1\alpha_2\alpha_5, \\
\alpha_2^{\alpha_0} &= \alpha_4\alpha_2\alpha_5\alpha_2\alpha_3\alpha_1, \\
\alpha_3^{\alpha_0} &= \alpha_1\alpha_3\alpha_5\alpha_3\alpha_2\alpha_4, \\
\alpha_4^{\alpha_0} &= \alpha_3\alpha_4\alpha_5\alpha_4\alpha_1\alpha_2, \\
\alpha_5^{\alpha_0} &= \alpha_2\alpha_5\alpha_3\alpha_5\alpha_1\alpha_4, \\
[\alpha_i, \alpha_j]^2 &= 1 \text{ whenever } i, j > 0, \\
[\alpha_4, \alpha_1] &= \alpha_5\alpha_2\alpha_1\alpha_4\alpha_3, \\
[\alpha_2, \alpha_1] &= \alpha_5\alpha_2\alpha_1\alpha_3\alpha_4.
\end{aligned}$$

5. Action on neighbours

In this section we examine the action of Q on the set $\Delta_1(a)$ of points that are adjacent to a in Δ. Recall that a is on five lines of $\mathcal{G}$, these are labelled by five distinct colours. By assumption, each of the lines is invariant under the action of Q. Therefore every element of Q either fixes the ith line pointwise, or it fixes a and interchanges the remaining two points on the line. Thus the induced action of Q on $\Delta_1(a)$ is a subgroup of the elementary abelian group of order 2^5.

Lemma 5.1. *The group induced by Q_0 on $\Delta_1(a)$ is elementary abelian of order 2^4. The group induced by Q has order 2^4 or 2^5 and none of $\alpha_1, \ldots, \alpha_5$ belongs to the Frattini subgroup of Q.*

Proof: Fix $i \in \{1, \ldots, 5\}$ and recall that the element α_i of Q_0 takes b to its neighbour c_i in Σ. Let M_i be the line containing b and c_i and let L_j be the line on a coloured j. Let d be the point on L_j closest to b and let e be the third point on M_i (other than b and c_i). If $j = i$, then there is a shortest path from b to a going through M_i and L_i. Clearly this shortest path goes through e and d, which implies that the Δ-distance between d and e is two. This in turn implies that the distance between d and c_i is three; so d is the point on L_i closest to c_i. Since α_i takes b to c_i and stabilizes L_i, it must fix d. Hence α_i fixes L_i pointwise.

Now assume $j \neq i$. We claim that the point d of L_j closest to b cannot be closest to c_i. Indeed, if it is, the distance from both b and c_i to d is three. By Lemma 2.1, M_i contains a unique point closest to d. This means that closest to d is the third point e and that the distance from d to e is two. However, in this case we have a shortest path from a to b via d and e, which means that M_i should have the same colour as L_j, a contradiction. We proved that d is not closest to c_i on L_j, which means that closest to d is the third point, say f. The element α_i stabilizes L_j and takes b to c_i. This yields that α_i takes d to f, that is, α_i acting on L_j fixes a and switches the other two points.

To summarize, each α_i fixes all three points of the line L_i and switches two points in each of the other four lines on a. In the action of Q on $\Delta_1(a)$, the elements α_i generate an elementary abelian group of order 2^4, whose elements switch points in an even number of lines on a.

Since the Frattini subgroup of Q is the smallest normal subgroup $\Phi(Q)$ such that $Q/\Phi(Q)$ is an elementary abelian 2-group, none of $\alpha_1, \ldots, \alpha_5$ belongs to $\Phi(Q)$. □

Note that any four of the involutions α_i generate the group induced by Q_0 on $\Delta_1(a)$. In fact, in this action the product of all α_i is the identity, and this is the only linear relation that the α_i satisfy.

Corollary 5.2. *The Frattini quotient of Q_0 is elementary abelian of order 2^4 or 2^5. The order is 2^4 if and only if $\alpha_1 \cdots \alpha_5$ (or the product in any other order) belongs to the Frattini subgroup $\Phi(Q_0)$.*

Proof: By the above, the rank of $Q_0/\Phi(Q_0)$ is at least four. On the other hand, since Q_0 is generated by five elements, the rank cannot be more than five. □

The structure of Σ reflects the different orders of $Q_0/\Phi(Q_0)$.

Lemma 5.3. *The graph Σ is bipartite if and only if $|Q_0/\Phi(Q_0)| = 2^5$.*

Proof: Clearly, Σ is bipartite if and only if every connected component is bipartite. Each connected component of Σ is isomorphic to $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$. Thus we can substitute Σ by this Cayley graph.

Let G be a group generated by a set of involutions I. It is well known that $\mathrm{Cay}(G, I)$ is bipartite if and only if G has an index two subgroup H that is disjoint from I. Indeed, if the graph is bipartite, then H is the stabilizer of the (unique, since the graph is connected) partition. Conversely, if such a subgroup H exists, then the two cosets of H are the parts of the partition.

If $\bar{Q}_0 = Q_0/\Phi(Q_0)$ has order 2^5, then the images $\bar{\alpha}_i$ of the five involutions α_i form a basis for $\bar{Q}_0$. We choose H as the full preimage in Q_0 of the subgroup in $\bar{Q}_0$ consisting of all elements having an even number of nonzero coordinates with respect to the basis $\{\bar{\alpha}_i\}$. Thus, the Cayley graph is bipartite.

If $\bar{Q}_0$ has order 2^4, then it follows from Lemma 5.1 that $\Phi(Q_0)$ is the kernel of the action of Q_0 on $\Delta_1(a)$. Note that every index 2 subgroup H in Q_0 contains $\Phi(Q_0)$. Since the product of all five α_i (in any order) lies in $\Phi(Q_0)$ (see Corollary 5.2), it also lies in H. But this implies that if four of the α_i lie outside of H, then the fifth lies in H. Thus, no such H exists in Q_0, so the Cayley graph is not bipartite. □

Example 5.4. Consider again the group Q and elements g and f from Examples 1.3, 3.4, and 4.7. In their conjugation action on Q, both g and f leave $\{\alpha_0, \ldots, \alpha_5\}$ invariant and act on the indices according to the permutations $(1, 2, 3, 4, 5)$ and $(1, 2, 4, 3)$, respectively. Observe that $\Phi(Q) = \Phi(Q_0)$ has order 32 and equals the commutator subgroups of both Q and Q_0.

6. Apartments and factorizations

An *apartment* in a generalized n-gon is a $2n$-cycle in the corresponding incidence graph Γ. In the collinearity graph Δ, it is a geometric n-cycle. Hence, in our context, an apartment is a geometric 8-cycle in Δ. Every such cycle is a union of two shortest paths connecting two opposite points. We choose a particular apartment C through the selected points a and b. It is the union of the two shortest paths between a and b passing through the lines on a carrying colours 1 and 5. We say that this apartment has colour $\{1, 5\}$. Clearly, for each pair of colours and each $c \in P_a$ there is a unique apartment through a and c coloured with that pair of colours. Note that every apartment on a is "coloured", and Q permutes these apartments and preserves colour.

We name all components of C: let it pass from a to point a_1 via line A_1 (labelled with colour 1), from a_1 to a_2 via line A_2, from a_2 to a_3 via A_3, from a_3 to b via A_4 (hence the intersection of A_4 with P_a is an edge of Σ of colour 1), from b to b_1 via line B_1 (its intersection with P_a has colour 5), from b_1 to b_2 via B_2, from b_2 to b_3 via B_3, and finally, from b_3 back to a via B_4, which is the line on a marked with colour 5. This notation is Δ-style, because it distinguishes between points and lines.

It is convenient to also use a more symmetric, Γ-style notation. Let $p_1 = A_1$, $p_2 = a_1$, $p_3 = A_2$, $p_4 = a_2$, $p_5 = A_3$, $p_6 = a_3$ and $p_7 = A_4$. Symmetrically, let $q_1 = B_1$, $q_2 = b_1$, $q_3 = B_2$, $q_4 = b_2$, $q_5 = B_3$, $q_6 = b_3$ and $q_7 = B_4$. For $i, j \in \{1, \ldots, 7\}$, define U_{ij} to be the stabilizer in Q of both p_i and q_j, and set $U_i = U_{ii}$.

For $i \in \{1, \ldots, 7\}$, the opposite vertices p_i and q_i of C, whether points or lines, are opposite. By definition, Q fixes the base point a, so U_i fixes every vertex on the half of C which is the shortest path between p_i and q_i passing through a. Hence U_i resembles what for Moufang generalized polygons is a *root subgroup* [16]. In Proposition 6.1(i) we show that it has the same order as a root group. However, we cannot prove (directly, without use of our main result) that U_i is a root group as this would require that it fix all vertices of Γ adjacent to a non-end vertex of the root (that is, the shortest path from p_i to q_i through a).

We also use a Δ-style notation for U_i that distinguishes between points and lines. For $i \in \{1, \dots, 4\}$, let S_i be the joint setwise stabilizer in Q of A_i and B_i. Then $S_i = U_{2i-1}$. Similarly, for $i \in \{1, 2, 3\}$, we let T_i be the stabilizer in Q of both a_i and b_i, so $T_i = U_{2i}$.

Proposition 6.1. *The groups S_i $(i = 1, 2, 3, 4)$ and T_i $(i = 1, 2, 3)$ satisfy the following properties.*

(i) Every S_i has order 2, every T_i has order 4.
(ii) If $i \leq j$, then $U_{ij} = U_i U_{i+1} \cdots U_j$. In particular, $Q = U_{17} = S_1 T_1 S_2 T_2 S_3 T_3 S_4$.
(iii) Every contiguous subproduct in the factorization of (ii) is a subgroup of Q.
(iv) Each element of Q can be uniquely written as $s_1 t_1 s_2 t_2 s_3 t_3 s_4$, where $s_i \in S_i$ and $t_i \in T_i$.

Proof: We claim that the vertex-wise stabilizer in Q of every simple path of length at least nine in Γ is trivial if this path contains a. It suffices to consider the case where the path has length exactly nine. Such a path lies in a unique apartment in Γ, which in turn contains a unique point from P_a. Since the action of Q on P_a is regular, the stabilizer of the path is trivial, proving the claim.

Note that U_{ij} stabilizes vertex-wise the path γ in Γ obtained by combining the shortest path from q_j to a and the shortest path from a to p_i. This combined path γ has length $8 + i - j$. Hence, if $j < i$ then the claim implies that $U_{ij} = 1$.

Suppose now that $i \leq j$. Let $\delta_i = 4$ if i is even (so p_i and q_i are points), and $\delta_i = 2$ otherwise. We claim that $|U_{ij}| = \delta_i \delta_{i+1} \cdots \delta_j$. Let N denote the right-side product. Note that the path γ extends in N ways to a simple path of length nine in Γ, and hence γ is contained in exactly N apartments, say $C_1, \dots, C_N$. We claim that the action of U_{ij} on the N apartments C_k is regular. Indeed, every C_k contains a unique point w_k from P_a. Since $U_{ij} \leq Q$, the stabilizer of w_k in U_{ij} is trivial, and hence also the stabilizer in U_{ij} of C_k is trivial. Now pick two apartments, C_k and $C_{k'}$. Note that both C_k and $C_{k'}$ pass through $p_1 = A_1$ and $q_7 = B_4$. Hence C_k and $C_{k'}$ have colour $\{1, 5\}$. It follows that $x \in Q$ taking w_k to $w_{k'}$ also takes C_k to $C_{k'}$. This element fixes all vertices in the intersection of C_k and $C_{k'}$. In particular, x fixes p_i and q_j and so $x \in U_{ij}$, proving that U_{ij} acts regularly on the N apartments C_k. This shows that $|U_{ij}| = N$. Since $U_i = U_{ii}$, it follows that $|U_i| = \delta_i$, which proves (i).

Next consider U_{ij} and $U_{i'j'}$, where $i \leq i'$, $j \leq j'$, and $i' \leq j + 1$. It follows from the definition that $U_{ij} \cap U_{i'j'} = U_{i'j}$. Both U_{ij} and $U_{i'j'}$ are subgroups of $U_{ij'}$, so $U_{ij} U_{i'j'} \subseteq U_{ij'}$. On the other hand,

$$|U_{ij} U_{i'j'}| = \frac{|U_{ij}| \cdot |U_{i'j'}|}{|U_{ij} \cap U_{i'j'}|} = \frac{|U_{ij}| \cdot |U_{i'j'}|}{|U_{i'j}|} = \frac{\delta_i \cdots \delta_j \delta_{i'} \cdots \delta_{j'}}{\delta_{i'} \cdots \delta_j} = \delta_i \cdots \delta_{j'} = |U_{ij'}|,$$

proving that $U_{ij} U_{i'j'} = U_{ij'}$. Applying this factorization consecutively, with $i < j$, we find $U_{ij} = U_{i,j-1} U_{jj} = \dots = U_{ii} U_{i+1,i+1} \cdots U_{jj} = U_i U_{i+1} \cdots U_j$. In particular, $Q = U_{17} = U_1 U_2 \cdots U_7$, which completes the proof of (ii) and (iii).

Since $\delta_1 \delta_2 \cdots \delta_7 = 2^{1+2+1+2+1+2+1} = 2^{10}$, the number of products $s_1 t_1 s_2 t_2 s_3 t_3 s_4$ coincides with $|Q|$, so (iv) follows. □

This factorization of Q has consequences for $\alpha_1, \dots, \alpha_5$ and $\mathrm{Cay}(Q, \{\alpha_1, \dots, \alpha_5\})$.

Lemma 6.2. *(i)* $S_1 = \langle \alpha_5 \rangle$ *and* $S_4 = \langle \alpha_1 \rangle$.

(ii) For each $i \in \{2, 3, 4\}$ *and* $x \in (U_1 \cdots U_i)^{\#} \cup (U_{8-i} \cdots U_7)^{\#}$*, the distance of* x *in* $\mathrm{Cay}(Q, \{\alpha_1, \ldots, \alpha_5\})$ *to* 1 *is at least* $8 - i$.

Proof: Observe α_1 takes b to the other point c_1 of the line A_4 that lies in P_a, so α_1 stabilizes A_4. Since Q stabilizes B_4, we deduce that $\alpha_1 \in S_4$, proving that $S_4 = \langle \alpha_1 \rangle$. Symmetrically, $S_1 = \langle \alpha_5 \rangle$. This implies (i).

As for (ii), if $x \in (U_1 \cdots U_i)^{\#}$, then x fixes the vertex q_i of Γ. But q_i has distance i to b in Γ, so b^x has the same distance to q_i. This implies that $d(b, b^x)$ is at most $2i$ in Γ and so at most i in Δ. The path from b to b^x via q_i does not lie in Σ and $i \leq 4$ and the geometric girth of Δ is eight, so the distance between b and b^x in Σ is at least $8 - i$. The conclusion of the statement now follows from Lemma 4.4. The argument for $x \in (U_{8-i} \cdots U_7)^{\#}$ is similar, as such an element fixes p_{8-i}. □

Lemma 6.3. *Suppose that* $|Q_0/\Phi(Q_0)| = 2^5$.

(i) If X *is the set of all vertices of* $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$ *at distance at least six from the vertex* 1*, then* X *has a subset of size five with all mutual distances in* $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$ *at least two.*

(ii) Assume that $|Q_0| \geq 2^9$*. If* X *is the set of all vertices of* $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$ *at distance at least seven from the vertex* 1*, then* X *has a subset of size five with all mutual distances in* $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$ *at least four.*

Proof: Instead of $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$, we use the isomorphic graph Σ_0, the connected component of Σ containing b. Since $|Q_0/\Phi(Q_0)| = 2^5$, this graph is bipartite by Lemma 5.3.

(i) Assume first that $|Q_0| = 2^8$. Let $X = \Sigma_{\geq 6}(b)$, the set of vertices of Σ_0 at distance at least six from b. We need to find a subset Z of X of size five so that all mutual distances in Z are at least two. Each colour i contributes one vertex z_i to Z. We describe this for $i = 1$ and then invoke similarity for all other colours.

Let $i = 1$. If $R = U_{57}$, then R is the setwise stabilizer in Q of the line A_3. By Proposition 6.1(ii), (iv), $R = S_3T_3S_4$ has order 16. Corollary 4.6 implies that $[Q : Q_0] \leq 4$, so $|R \cap Q_0| \geq 4$. Since $S_4 = \langle \alpha_1 \rangle \leq Q_0$ by Lemma 6.2, $R \cap Q_0 > S_4$ and so we can select $x \in (R \cap Q_0) \setminus S_4$. Let e be the edge $\{b, b^{\alpha_1}\} = \{b, c_1\}$ (this edge is the intersection of A_4 with P_a) and let $f = e^x$. Since x is not in S_4, we deduce that $A_4^x \neq A_4$. Hence the distance between every vertex on e and every vertex on f in Δ is at most three; furthermore, the shortest path goes via the lines A_4, A_3, and A_4^x. Since the geometric girth of Δ is eight, the distance in Σ_0 between e and f is at least five.

Since Σ_0 is bipartite, f contains a vertex z_1 at distance at least six from b. Hence $z_1 \in X$. Similarly, construct z_i for each colour $i \geq 2$ (notice that Lemma 6.2 holds for other choices than 1 and 5 with the indices suitably permuted).

Consider two of these vertices, z_i and z_j; since each is connected to b via a path in Δ of length three, there is a path of length six in Δ going from z_i via b to z_j and this simple path has no edges from Σ. Therefore the distance in Σ_0 between z_i and z_j is at least two, proving (i).

(ii) Assume now that $|Q_0| = 2^{10-d}$ for $d \in \{0, 1\}$. Let X be the set of vertices of Σ_0 at distance at least seven from b, and take $R = U_{67}$, the stabilizer in Q of the point a_3. Note that $R = T_3S_4$ and so $|R| = 8$. Thus $|R \cap Q_0| \geq 2^{3-d} \geq 4$, since $[Q : Q_0] = 2^d$. This again allows us to choose $x \in (R \cap Q_0) \setminus S_4$. As above, let $e = \{b, c_1\}$ and $f = e^x$. Since $A_4^x \neq A_4$ and $a_3 \in A_4$ is stabilized by x, so $a_3 \in A_4 \cap A_4^x$, the distance in Δ between every vertex on e and every vertex on f is two and, furthermore, the shortest path goes via a_3. Thus the distance between e and f in Σ_0 is at least six.

Since Σ_0 is bipartite, we find a vertex z_1 on f such that the distance in Σ_0 between b and z_1 is at least seven, so $z_1 \in X$. By similarity, select z_i for each colour i.

If $i \neq j$, then Δ contains a simple path of length four going from z_i via b to z_j. Consequently, the distance in Δ between z_i and z_j is four, so, by Lemma 2.1, their mutual distance in Σ_0 is at least four. □

Recall that the *distance* between a pair of edges is the minimum of the four distances between the end vertices of the edges.

Lemma 6.4. *Suppose that $|Q_0/\Phi(Q_0)| = 2^4$ and $|Q_0| \geq 2^9$. Let X be the set of all vertices x of $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$ which are at distance at least six from the vertex 1, and such that $x^2 = 1$. Let (X, E) be the subgraph of $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$ induced on X. Then E has a subset $S = \{f_1, \ldots, f_5\}$, where f_i has colour i, satisfying the following properties.*

(i) The distance between every pair of edges in S is at least four.
(ii) If $f_i = \{x_i, y_i\}$ then x_i and y_i commute with α_i in Q_0 and they do not commute with α_j when $j \neq i$.

Proof: Once again we work in Σ_0 instead of $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$. We exploit the same idea as in Lemma 6.3, except this time Σ_0 is not bipartite. We provide complete details only for f_1 and select all other f_i by similarity.

Let $e = \{b, c_1\}$, where $c_1 = b^{\alpha_1}$, and set $R = U_{67} = T_3S_4$. Since $[Q : Q_0] \leq 2$, $|S_4| = 2$, and $|R| = 8$, we deduce that $R \cap Q_0 > S_4$. Next we select $x \in (R \cap Q_0) \setminus S_4$, but we must ensure that $x^2 = 1$ and x commutes with α_1 (recall $\langle \alpha_1 \rangle = S_4$ from Lemma 6.2).

Since $R = T_3S_4$ and $R \cap Q_0 > S_4$, we can write $R \cap Q_0 = (T_3 \cap Q_0)S_4$. Since $T_3 \cap Q_0$ has index 2 in $R \cap Q_0$, it is a normal subgroup. In particular, $T_3 \cap Q_0$ contains an involution that is central in $R \cap Q_0$. Choose this element as x. Clearly, $x^2 = 1$ and x commutes with α_1. Also $x \in T_3$ and so $x \notin S_4$ in view of Proposition 6.1(iv).

Let $f_1 = e^x$. Note that $f_1 = e^x = \{b, b^{\alpha_1}\}^x = \{b^x, b^{\alpha_1 x}\}$. Hence, using the map ϕ of Lemma 4.4, we can take x_1 and y_1, as in (ii), to be x and $\alpha_1 x$. Clearly, both x_1 and y_1 are involutions and commute with α_1.

Manifestly, every shortest path between the vertices on e and f_1 has length two and passes through a_3. This implies that both vertices of f_1 lie in the set of all vertices of Σ_0 that are at distance at least six from b.

This also implies that $x = x_1$ does not commute with α_j for all $j \neq 1$. We set $z = b^x$. Assuming that $x\alpha_j = \alpha_j x$, we obtain that $c_j^x = b^{\alpha_j x} = b^{x\alpha_j} = z^{\alpha_j}$. Since there is a path in Δ of length two connecting b and z via a_3 and the colour 1 lines A_4 and A_4^x, we deduce that $c_j = b^{\alpha_j}$ and z^{α_j} are connected by a path of length two in Δ involving two lines of colour 1. On the other hand, b and c_j are connected by an edge of colour j in Σ_0, so $z = b^x$ and $z^{\alpha_j} = c_j^x$ are also connected in Σ_0 by an edge of colour j. This gives a geometric cycle in Δ of length six, which is a contradiction. Hence x_1 and α_j cannot commute. The argument for $y = y_1$ is similar if we set $z = b^y$ instead of b^x. Hence y_1 and α_j do not commute.

We select f_i similarly for $i \geq 2$. It remains to check the distances between different f_i. Taking vertices u on f_i and v on f_j, we observe as in Lemma 6.3 that there is a simple path of length four in Δ going from u to v via b. Therefore the distance in Σ_0 between u and v is at least four. □

Example 6.5. The groups U_i, for $1 \leq i \leq 7$, defined at the beginning of this section for the colours $\{1, 5\}$, coincide with $\langle u_{9-i} \rangle$ defined in Example 1.3.

To find easily computable necessary conditions for groups Q of order 2^{10} to appear in the conclusion of Theorem 1.2, we study how factorizations of $W = U_{26} = T_1 S_2 T_2 S_3 T_3$ can be extended to factorizations of Q.

Lemma 6.6. *The group W has order 2^8 and is normal in Q, and $Q/W \cong 2^2$.*

Proof: The claim about the order of W follows from Proposition 6.1. Since W has index 2 in each of $S_1 W$ and $W S_4$, it is normal in both and also in $S_1 W S_4 = Q$. Recall from Lemma 6.2 that $S_1 = \langle \alpha_5 \rangle$ and $S_4 = \langle \alpha_1 \rangle$. Thus, Q/W has order 4 and is generated by two involutions, so it is elementary abelian. □

Definition 6.7. Let G be a 2-group with a collection of subgroups $H_1, \ldots, H_k$ such that (1) $G = H_1 H_2 \ldots H_k$; (2) $|G| = |H_1| \cdot |H_2| \cdots |H_k|$; and (3) for all $i \leq j$, the product $H_i H_{i+1} \cdots H_j$ is a subgroup of G. Then $H_1, \ldots, H_k$ form a *tight factorization* of G.

The factorizations of Q in Proposition 6.1(ii) are tight.

Given two factorizations $G = H_1 \cdots H_k$ and $G' = H'_1 \cdots H'_k$ with the same number k of factors, an isomorphism between the two factorizations is a group isomorphism $\psi : G \to G'$ such that $\psi(H_i) = H'_i$ for all i. Clearly, for all $i \leq j$, the map ψ induces an isomorphism of $H_i \cdots H_j$ onto $H'_i \cdots H'_j$.

Define the *left automorphism group* $\mathrm{Aut}^-(G; H_1, \ldots H_k)$ of the factorization to be the group of automorphisms of G normalizing each subproduct subgroup $H_1 \cdots H_i$ for $i = 1, \ldots, k$. Similarly, the *right automorphism group* $\mathrm{Aut}^+(G; H_1, \ldots, H_k)$ consists of all automorphisms of G normalizing each subproduct subgroup $H_i \cdots H_k$ for $i = 1, \ldots, k$. Let $\mathrm{Aut}(G; H_1, \ldots, H_k)$ be those automorphisms of G normalizing each H_i. Clearly, $\mathrm{Aut}(G; H_1, \ldots, H_k) = \mathrm{Aut}^-(G; H_1, \ldots, H_k) \cap \mathrm{Aut}^+(G; H_1, \ldots, H_k)$.

Recall that W is normal in $Q = S_1 W S_4$ with $S_1 = \langle \alpha_5 \rangle$ and $S_4 = \langle \alpha_1 \rangle$. Observe that $U_{2j} = U_2 \cdots U_j$ has index 2 in $U_{1j} = U_1 U_2 \cdots U_j$. Hence $\alpha_5 \in S_1 = U_1$ normalizes each subgroup $U_2 \cdots U_j$. Similarly, $\alpha_1 \in S_4 = U_7$ normalizes each subgroup $U_i \cdots U_6$.

Lemma 6.8. *Let $L = \mathrm{Aut}^-(W; T_1, S_2, T_2, S_3, T_3)$ and $R = \mathrm{Aut}^+(W; T_1, S_2, T_2, S_3, T_3)$. The involution α_5 induces an automorphism of W contained in L and, symmetrically, α_1 induces an automorphism of W contained in R.*

We now investigate restrictions on the elements α_i for $i = 2, 3, 4$.

Lemma 6.9. *For each $i \in \{2, 3, 4\}$ there exists $w_i \in W \setminus (T_1S_2T_2 \cup T_2S_3T_3 \cup \Phi(Q_0))$ such that $\alpha_i = \alpha_5 w_i \alpha_1$.*

Proof: We first show $\alpha_i \in \alpha_5 W \alpha_1$. Since $Q = S_1WS_4$, each $x \in Q$ is one (and only one) of the following types: $x = w$, $\alpha_5 w$, $w\alpha_1$, or $\alpha_5 w \alpha_1$ for some $w \in W$. Therefore it suffices to show that α_i does not belong to $WS_4 \cup S_1W$. By Lemma 5.1 each α_i fixes all three points on the line containing a that is marked with colour i, and moves two points on each of the other four lines on a. In particular, α_i does not lie in $WS_4 = U_{27}$, which is the pointwise stabilizer in Q of the line A_1 marked with colour 1. Similarly, α_i is not contained in $S_1W = U_{16}$, which is the pointwise stabilizer of the line B_4 of colour 5.

Since $\alpha_i = \alpha_5 w_i \alpha_1$ for some $w_i \in W$, we deduce that $w_i = \alpha_5 \alpha_i \alpha_1$ is at distance three to 1 in $\mathrm{Cay}(Q_0, \{\alpha_1, \dots, \alpha_5\})$. By Lemma 6.2(ii) and Corollary 5.2, respectively, this implies $w_i \notin T_1S_2T_2 \cup T_2S_3T_3$ and $w_i \notin \Phi(Q_0)$. □

Lemma 6.10. *The element $r = (\alpha_5\alpha_1)^2 = [\alpha_5, \alpha_1]$ of W satisfies the following:*

(i) $r = w_i w_i^{\alpha_5\alpha_1}$ *for* $i \in \{2, 3, 4\}$*;*
(ii) $r^{\alpha_5} = r^{\alpha_1} = r^{-1}$*;*
(iii) $r \in \Phi(Q_0) \setminus (T_1S_2 \cup S_3T_3)$.

Proof: Recall that each α_i has order 2. Since $\alpha_i^2 = (\alpha_5 w_i \alpha_1)^2 = \alpha_5 w_i \alpha_1 \alpha_5 w_i \alpha_1$, we deduce that $\alpha_5 w_i \alpha_1 \alpha_5 w_i \alpha_1 = 1$, so $w_i \alpha_1 \alpha_5 w_i \alpha_1 \alpha_5 = 1$. Multiplying both sides on the right with $r = (\alpha_5\alpha_1)^2$ and cancelling $\alpha_1\alpha_5\alpha_5\alpha_1$ establishes (i). Part (ii) follows from the definition of r. Part (iii) follows from Lemma 6.2(ii), Corollary 5.2, and the fact that r is at distance four to 1 in $\mathrm{Cay}(Q_0, \{\alpha_1, \dots, \alpha_5\})$. □

Lemma 6.11. $\Phi(Q) = \Phi(W)[\alpha_5, W][\alpha_1, W]$.

Proof: Clearly, the right hand side is contained in $\Phi(Q)$, so we only need to establish the reverse inclusion. Note that $X = \Phi(W)[\alpha_5, W][\alpha_1, W]$ is normal in W and also invariant under α_5 and α_1, since it contains $[\alpha_5, W]$ and $[\alpha_1, W]$, respectively. In particular, X is normal in Q. Hence we just need to verify that $\bar{Q} = Q/X$ is elementary abelian. Observe that α_5 and α_1 have order 2, and all nontrivial elements of W/X have order 2 since $X \geq \Phi(W)$. Hence it suffices to show that $\bar{Q}$ is abelian. Clearly, W/X is abelian and also the images $\bar{\alpha}_5$ and $\bar{\alpha}_1$ of α_5 and α_1 modulo X centralize W/X. To show that $\bar{\alpha}_5$ and $\bar{\alpha}_1$ commute, we prove that $r \in X$. By Lemma 6.10, $r = w_i w_i^{\alpha_5\alpha_1} = w_i^2[w_i, \alpha_5\alpha_1]$. Since each of $\bar{\alpha}_5$ and $\bar{\alpha}_1$ centralize W/X, the product $\bar{\alpha}_5\bar{\alpha}_1$ centralizes W/X. Therefore $\bar{r} = \bar{w}_i^2[\bar{w}_i, \bar{\alpha}_5\bar{\alpha}_1] = \bar{w}_i^2 = 1$, since W/X is elementary abelian. □

Recall from Corollary 5.2 that $Q_0/\Phi(Q_0)$ is isomorphic to 2^4 or 2^5. Since $Q/W \cong 2^2$ and $\Phi(Q) = \Phi(W)[\alpha_5, W][\alpha_1, W] \leq W$, we obtain the following result when Σ has a single connected component.

Corollary 6.12. *If Q_0 has order 2^{10}, then $W/\Phi(W)[\alpha_5, W][\alpha_1, W]$ is isomorphic to 2^2 or 2^3.*

This nontrivial condition on W cannot be verified without first knowing the automorphisms induced by α_5 and α_1. Hence we need a weaker form that can be verified directly from the factorization $W = T_1S_2T_2S_3T_3$. The automorphisms of W induced by α_5 and α_1 are in L and R, respectively. The following is a consequence of Corollary 6.12 and the fact that both α_5 and α_1 are involutions.

Corollary 6.13. *If Q_0 has order 2^{10}, then $W/\Phi(W)[L, W][R, W]$ is elementary abelian of rank at most 3.*

Proof: The automorphisms induced by α_5 and α_1 lie in L and R, respectively, so $\Phi(W)[\alpha_5, W][\alpha_1, W] \leq \Phi(W)[L, W][R, W]$. □

Lemma 6.14. *The elements w_2, w_3, w_4 satisfy the following properties:*

(i) $|w_i| > 2$;
(ii) $w_i^{\alpha_5}w_i, w_i^{\alpha_1}w_i$ *lie outside* $T_1S_2 \cup S_3T_3$;
(iii) $|w_i^{-1}w_j| > 2$ *if* $i \neq j$;
(iv) $|Q_0/\Phi(Q_0)| = 2^4$ *if and only if* $w_2w_3w_4 \in \Phi(Q_0)$;
(v) $w_3 \neq w_4w_2$.

Proof: (i) The square of $w_i = \alpha_5\alpha_i\alpha_1$ is at distance at most six to 1 in $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$, which has girth at least eight, so $|w_i| \geq 4$.
(ii) $w_i^{\alpha_5}w_i = \alpha_i\alpha_1\alpha_i\alpha_1$ is at distance four to 1 in $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$, so does not belong to $T_1S_2 \cup S_3T_3$ by Lemma 6.2.
Claim (iii) follows from the fact that $w_i^{-1}w_j$ is conjugate to $\alpha_i\alpha_j$.
(iv) Corollary 5.2 implies that $\alpha_5\alpha_2\alpha_3\alpha_4\alpha_1 \in \Phi(Q_0)$ if and only if $|Q_0/\Phi(Q_0)| = 2^4$. The assertion follows from $w_2w_3^{-1}w_4 = \alpha_5\alpha_2\alpha_3\alpha_4\alpha_1$ and $w_3^2 \in \Phi(Q_0)$.
(v) This is immediate from the previous computation as $\alpha_5\alpha_2\alpha_3\alpha_4\alpha_1 = 1$ would contradict the girth of Σ being at least eight. □

7. The groups of order dividing 2^9

Corollary 4.6 states that $Q_0 = \langle \alpha_1, \ldots, \alpha_5 \rangle$ is a group of order 2^n for $8 \leq n \leq 10$. We now prove Theorem 1.2 when the order of Q_0 divides 2^9 (or, equivalently, Σ is disconnected).

Definition 7.1. (G, I) is an *involution pair* if I is a set of involutions which generates a 2-group G, and $\mathrm{Cay}(G, I)$ has girth at least eight.

Observe that if (G, I) is an involution pair, then, for every subset S of I, the pair $(\langle S \rangle, S)$ is an involution pair.

Corollary 4.5 establishes that each of $\mathrm{Cay}(Q, \{\alpha_1, \ldots, \alpha_5\})$ and $\mathrm{Cay}(Q_0, \{\alpha_1, \ldots, \alpha_5\})$ has girth at least eight. Corollary 5.2 implies that $|Q_0/\Phi(Q_0)| = 2^k$ where $k \in \{4, 5\}$. Lemma 5.1 and the observation that the Frattini subgroup of any subgroup (here, $\Phi(Q_0)$) is contained in Frattini subgroup of its overgroup (here, $\Phi(Q)$) show that none of the α_i is in $\Phi(Q_0)$.

We use these properties to obtain a list of candidate groups for Q_0.

Theorem 7.2. *Consider a group G of order 2^8 or 2^9 that satisfies the following:*

(i) $|G/\Phi(G)|$ is 2^4 or 2^5;

(ii) G has a generating set I of size 5 such that (G, I) is an involution pair.

There are 14 such groups of order 2^8, all satisfying $|G/\Phi(G)| = 2^5$. There are 421 and 32555 such groups of order 2^9, satisfying $|G/\Phi(G)| = 2^4$ and 2^5, respectively.

Theorem 7.2 was proved by investigating the relevant groups of order dividing 2^9; these are in the SMALLGROUPS library [2], distributed with GAP [1] and MAGMA [4].

We can easily determine those groups G which satisfy condition (i) of Theorem 7.2. The number of such groups is recorded in Table 2.

Order	$\|G/\Phi(G)\|$	Number
2^8	2^4	20 241
2^8	2^5	28 653
2^9	2^4	359 611
2^9	2^5	7 111 878

TABLE 2. Number of groups of order 2^8 and 2^9 satisfying Theorem 7.2(i)

The algorithm used to determine if G satisfies (ii) is the following.

(1) Determine the set J of involutions of G which lie outside $\Phi(G)$.
(2) Construct a list $\mathcal{M}_3$ of all G-automorphism class representatives of 3-element subsets of J.
(3) Construct the list $\mathcal{L}_3$ of all X in $\mathcal{M}_3$ such that $(\langle X\rangle, X)$ is an involution pair.
(4) For each $X \in \mathcal{L}_3$, choose $\alpha_4 \in J$ to obtain $Y = X \cup \{\alpha_4\}$ of size 4; decide if $(\langle S\rangle, S)$ is an involution pair for every $S \subseteq Y$ of cardinality 3 or 4. Record those Y which satisfy this condition to obtain $\mathcal{L}_4$.
(5) For each $X \in \mathcal{L}_4$, choose $\alpha_5 \in J$ to obtain $Y = X \cup \{\alpha_5\}$ of size 5; decide if $(\langle S\rangle, S)$ is an involution pair for every $S \subset Y$ of cardinality 3 or 4. Record those Y which satisfy this condition to obtain $\mathcal{L}_5$.
(6) For each $I \in \mathcal{L}_5$, decide if $\mathrm{Cay}(G, I)$ has girth at least eight. If not, then G does not satisfy condition (ii) of Theorem 7.2, and we remove I from the list $\mathcal{L}_5$.
(7) Return $\mathcal{L}_5$.

We now use additional properties of Q_0 to eliminate all but one of the groups of Theorem 7.2.

Proposition 7.3. *Among the groups G appearing in Theorem 7.2,*

(i) none of the 14 groups of order 2^8 and none of the 32555 groups of order 2^9 with $|G/\Phi(G)| = 2^5$ satisfies Lemma 6.3;

(ii) precisely two of the 421 *groups of order* 2^9 *with* $|G/\Phi(G)| = 2^4$ *satisfy condition (i) of Lemma* 6.4*: namely,* $512\#233888$ *and* $512\#384204$*;*
(iii) $512\#233888$ *does not satisfy condition (ii) of Lemma 6.4.*

Hence Q_0 is uniquely determined as $512\#384204$; of course, this is the group of Example 4.7.

Proposition 7.4. *Up to isomorphism, there is a unique generalized octagon of order* $(2,4)$ *satisfying the condition of Theorem 1.2 in which* Q_0 *is as in Example* 4.7*.*

Proof: Assume that $\mathcal{G}$ is a generalized octagon as in the hypothesis. Straightforward calculations show that, up to automorphism classes of Q_0, there is, up to ordering of the indices, a unique 5-tuple $\alpha_1, \dots, \alpha_5$ such that the Cayley graph $\mathrm{Cay}(Q_0, \{\alpha_1, \dots, \alpha_5\})$ satisfies the conclusions of Lemma 6.4. Consequently, we may assume, without loss of generality, that $\alpha_1, \dots, \alpha_5$ are as given in Example 4.7. This means that Σ, the graph induced by Δ on the set of points P_a opposite to a, being the disjoint union of two copies of $\mathrm{Cay}(Q_0, \{\alpha_1, \dots, \alpha_5\})$, is the same as in the known octagon of order $(2,4)$. Let Σ_0 be the component of Σ containing the vertex b used to define the involutions α_i. Let Σ_1 be the other connected component of Σ.

Let P_0 be the vertex set of Σ_0. We first determine the Δ-distance on P_0. In view of transitivity of Q_0 on P_0, it suffices to find the distances from b to all other points c. Clearly, if the distance between b and c in Σ is at most 4, then their distance in Δ is the same. As Σ_0 has diameter 6, we may assume that c is at Σ-distance 5 or 6 to b. Now c has Δ-distance 2, 3, or 4 to b, and the shortest path from c to b in Δ goes via $\Delta_3(a)$.

Suppose that the Δ-distance between b and c is 2. This means that b and c are both collinear to a point $d \in \Delta_3(a)$. Let $\{b, b', d\}$ be the line through b and d and let $\{c, c', d\}$ be the line through c and d. Since the geometric girth of Δ is 8, the edges $\{b, b'\}$ and $\{c, c'\}$ of Σ must be at distance 6. For every edge of Σ there are exactly four vertices at distance 6 from that edge, and on these vertices there is only one edge. It follows that there are exactly $2 \cdot 5 = 10$ points in P_0 that are at Δ-distance 2 from b, but not at distance 2 in Σ.

The above observation also implies that every $d \in \Delta_3(a)$ lies on at most two lines meeting Σ_0. Since this equally applies to Σ_1 and d lies on a total of four lines meeting Σ, we conclude that d lies on exactly two lines meeting Σ_0 and exactly two lines meeting Σ_1. This allows us to claim that there is a bijection between $\Delta_3(a)$ and the collection of all 4-sets in P_0 formed by two edges at distance 6. We call such 4-sets *double edges*. The stabilizer in Q_0 of a double edge D has size 4 and acts regularly on D. We have shown that the points of P_0 at distance 2 from b can be identified uniquely.

This also gives us some information about points at distance 3 to b, namely, all points c such that either there exists a point b' adjacent to b that is at distance 2 from c, or symmetrically, there exists a point c' adjacent to c that is at distance 2 from b. Clearly, all such points c can be identified. If c is not of this kind and at distance 3 to b in Δ, then the unique shortest path from b to c goes via adjacent points d and d' of $\Delta_3(a)$. Let D and D' be the double edges on b and c, corresponding to d and d', respectively.

We claim that the stabilizer in Q_0 of $D \cup D'$ coincides with the stabilizer of the edge $\{d, d'\}$ and has order 8. In view of the regularity of Q_0 on P_0, it suffices to show that every member g of Q_0 mapping b to an element of $D \cup D'$ preserves all of $D \cup D'$. The third point, say e, on the line containing d and d' is collinear with a unique point, say f, in $\Delta_1(a)$. This implies that all four edges of $D \cup D'$ in Σ_0 have the same colour. In particular, g maps the edge $\{b, b'\}$ on b in D to an edge of the same colour, which forces the third point on the line containing b^g and $(b')^g$ to be d or d'. Since all points of $D \cup D'$ have Δ-distance 3 to f and 4 to a, the element g fixes f, and hence also e. This implies that g stabilizes $\{d, d'\}$, and, because of the bijective correspondence mentioned above, also $D \cup D'$. This settles the claim. Observe that the distance between D and D' in Σ is at least 5.

The claim allows us to identify all pairs (b, c) at distance 3. Namely, the set of all points from P_0 at Σ-distance ≥ 5 to D consists of exactly 12 points and it is a disjoint union of three double edges D'. Only one of these sets D' satisfies the property that the stabilizer in Q_0 of $D \cup D'$ is of size 8 (it is of size 4 for the other two double edges). Hence we can uniquely identify all points in P_0 at Δ-distance 3 from b. As the only remaining Δ-distance is 4, the Δ-distance between any pair of points from P_0 is uniquely determined (and, hence, as in Example 4.7).

To show that knowledge of Σ_0 and the Δ-distance on P_0 determine Δ uniquely, we change our language. The graph Δ, being distance regular, can be realized by unit vectors in a Euclidean space (an eigenspace of its adjacency matrix) in such a way that all mutual inner products only depend on the distances of the corresponding vertices in Δ. See [5, Proposition 4.4.1] for details. In particular, Δ can be realized by 1755 unit vectors e_x $(x \in P)$ in a 78-dimensional Euclidean space in such a way that $(e_x, e_y) = (-\frac{1}{2})^k$, where k is the Δ-distance between x and y.

We know all distances on the set P_0. This gives us the Gram matrix of size 512×512. The rank of this matrix is exactly 78. Hence we have found the unique (up to an isometry) realization of P_0 by unit vectors. If $\{x, y, z\}$ is a line of Δ then, by an easy calculation, the vector $e_x + e_y + e_z$ has length zero. Hence $e_x + e_y + e_z = 0$. Since every $d \in \Delta_3(a)$ lies on a line $\{d, c, c'\}$, where c and c' are adjacent vertices of Σ_0, we can now construct all vectors e_d as $-e_c - e_{c'}$ for all edges $\{c, c'\}$ of Σ_0. The above inner product formula provides the complete information about the Δ-distances among the vertices of $\Delta_3(a)$. Taking all adjacent vertices in $\Delta_3(a)$ and repeating the above argument, we construct all vectors e_d for $d \in \Delta_2(a)$, also in $\Delta_1(a)$, and finally $\Delta_0(a) = \{a\}$.

Hence we have uniquely recovered the system of unit vectors representing the points in $P \setminus P_1$, where $P_1 = P_a \setminus P_0$ is the vertex set of Σ_1. It remains to recover P_1.

To this end, we define the *purple graph* on $\Delta_3(a)$, whose edges are all pairs $\{d, d'\}$ such that $(e_d, e_{d'}) = \frac{1}{4}$ and the distance between d and d' in the subgraph of Δ induced on $P \setminus P_1$ is not 2. Clearly, $\{d, d'\}$ is an edge of the purple graph if and only if there is a (unique) point x in P_1 collinear with both d and d'.

Since the geometric girth of Δ is 8, the three edges in every 3-clique of the purple graph must correspond to the same $x \in P_1$. Every $x \in P_1$ is collinear with exactly five points from $\Delta_3(a)$. It follows that the maximal cliques in the purple graph have size 5 and they bijectively correspond to the points from P_1. Furthermore, different maximal cliques meet trivially or in one point.

Now that we can identify points of P_1 with maximal cliques of the purple graph, we can decide if two points of P_1 are adjacent: two points x and x' of P_1 are adjacent if and only if the corresponding maximal cliques X and X' of the purple graph meet in one point d (the third point on the line through x and x') and additionally all Δ-distances between points in $X \setminus \{d\}$ and points in $X' \setminus \{d\}$ are 3. Since these distances can be read off from the known vectors representing points in $\Delta_3(a)$, this shows that the collinearity graph Δ is unique. □

This establishes Theorem 1.2 in the case where $|Q_0| \leq 2^9$. We finish this section with some remarks on the computations. These were performed using MAGMA. The SMALLGROUPS library provides a function to identify a group of order dividing 2^8. As a preprocessing step, we applied (an obvious variation of) the algorithm used in Theorem 7.2 to determine all involution pairs (G, I) where $|G|$ divides 256 and $3 \leq |I| \leq 4$. We record the number of such groups in Table 3. Hence we could readily decide in steps (4)–(5) whether or not $(\langle S \rangle, S)$ is an involution pair. The automorphism group of a 2-group was computed using the algorithm of [9]. The memory resources used to establish the result are small, but the time taken is significant: an estimate is approximately 10 CPU years running MAGMA 2.15. Most of this was used to prove Theorem 7.2.

Order	$\#I$	Number
2^6	3	11
2^7	3	33
2^7	4	20
2^8	3	124
2^8	4	539

TABLE 3. Number of groups of order dividing 2^8 arising in involution pairs

8. The groups of order 2^{10}

We now finish the proof of Theorem 1.2 by showing that the assumption $|Q_0| = 2^{10}$ leads to a contradiction.

There are $49\,487\,365\,422$ groups of order 2^{10} (see [2]), too many for the methods of Section 7 to be feasible. Our approach is based on Proposition 6.1: namely, $Q = Q_0$ admits a tight factorization $Q = S_1T_1S_2T_2S_3T_3S_4$ with the property that every factor S_i has order two and every factor T_i has order four. We call such a factorization a *2424242-factorization*, and similarly for subproducts. However, the number of tight factorizations of Q is also prohibitively large. Therefore we focus instead on the 'middle' of Q, the tight 42424-factorization of $W = U_{26} = T_1S_2T_2S_3T_3$ as in Section 6.

Using a standard extension algorithm, all tight 42424-factorizations of groups of order 256 were constructed using GAP. These factorizations were also determined directly by processing each relevant group of order 256 in MAGMA. This led to the following result.

Lemma 8.1. *Exactly* 3090 *of the* 56092 *groups of order* 256 *have at least one tight 42424-factorization.*

The 3090 groups have $1\,948\,859$ tight 42424-factorizations, representing all possibilities for $W = T_1S_2T_2S_3T_3$. For each factorization, an algorithm described below was applied in GAP that produces a list of all 5-tuples corresponding to a generating set $\alpha_1, \ldots, \alpha_5$ of an overgroup Q of W and to an extension of $T_1S_2T_2S_3T_3$ to a tight 2424242-factorization $S_1T_1S_2T_2S_3T_3S_4$ satisfying the necessary conditions of Sections 4 and 6 for Q to be as in the hypotheses of Theorem 1.2.

Let F_k be the set of all words of length at most k in the symbols $\alpha_1, \ldots, \alpha_5$ without repetitions. A member of F_k is 1-*balanced* if the sum of the occurrences of α_j over all $j \neq 5$ is even, and 5-*balanced* if the sum of the occurrences of α_j over all $j \neq 1$ is even. Observe that 1-balanced or 5-balanced words can be evaluated to elements of W by substituting α_j by $\alpha_5 w_j \alpha_1$ for $j = 2, 3, 4$ and interpreting α_1 and α_5 as automorphisms of W. (Note that Lemma 6.10 is needed here.) For a nonempty list S of elements from W of length at most 3, we define $E_k(S)$ to be the set of all evaluations of words in F_k as above with w_j (for $j \in \{2, 3, 4\}$) the $(j-1)$-st element of S. In our description below, we call the check that $E_7(S)$ does not contain the identity the *girth test* on S. This is justified by Lemma 3.2 or Corollary 4.5. If a triple w_2, w_3, w_4 passes this test, then the girth of $\mathrm{Cay}(Q, \{\alpha_1, \ldots, \alpha_5\})$ is at least eight because the products of α_i of length at most seven that are not tested have an odd number of α_1 or of α_5 and so are non-trivial.

Similarly, we use Lemma 6.2 to check (using computations in W only) that, for a 5-tuple $(a_5, a_1, w_2, w_3, w_4)$, where a_i is the automorphism of W induced by α_i, the distance conditions in $\mathrm{Cay}(G, \{\alpha_1, \ldots, \alpha_5\})$ on elements of $(U_1 \cdots U_i)^{\#} \cup (U_{8-i} \cdots U_7)^{\#}$, for $i \in \{2, 3, 4\}$, are satisfied. This check is called the *distance test*.

Let W, a group of order 2^8, have tight factorization $T_1S_2T_2S_3T_3$. The algorithm to process the factorization is the following.

(1) Compute $L = \mathrm{Aut}^{-}(W; T_1, S_2, T_2, S_3, T_3)$, $R = \mathrm{Aut}^{+}(W; T_1, S_2, T_2, S_3, T_3)$ and initialize $\mathcal{L}$ as the empty list.
(2) Compute $F = \Phi(W)$ and the subgroup $X = F[L, W][R, W]$ of W. If $|W/X| > 2^3$, then return $\mathcal{L}$ (in view of Corollary 6.13).
(3) Compute the sets $\mathcal{W}$ of all elements of $W \setminus (T_1S_2T_2 \cup T_2S_3T_3 \cup F)$ of order at least 4 and $\mathcal{R} = X \setminus (T_1S_2 \cup S_3T_3)$. If $|\mathcal{W}| < 3$, then return $\mathcal{L}$ (in view of Lemmas 6.9 and 6.14). If $\mathcal{R} = \emptyset$, then return $\mathcal{L}$ (in view of Lemma 6.10(iii)).
(4) Compute the list $\mathcal{C}_L$ of conjugacy classes of elements x of order at most two in L such that $|W/\langle F, [W, x]\rangle| \geq 4$, and, similarly, $\mathcal{C}_R$ with R instead of L. If $\mathcal{C}_L = \emptyset$ or $\mathcal{C}_R = \emptyset$ then return $\mathcal{L}$ (in view of Corollary 6.12).
(5) Compute $A = L \cap R$ and compile a list $\mathcal{X}_5$ of A-class representatives in the union of all conjugacy classes in $\mathcal{C}_L$.
(6) Compute the union $\mathcal{X}_1$ of all conjugacy classes in $\mathcal{C}_R$.
(7) For each automorphism $a_5 \in \mathcal{X}_5$:
    i. Compute $\mathcal{R}_5 = \{r \in \mathcal{R} \mid r^{a_5} = r^{-1}\}$ and $\mathcal{W}_5 = \{w \in \mathcal{W} \setminus F[W, a_5] \mid w^{a_5}w \notin T_1S_2 \cup S_3T_3\}$. If $\mathcal{R}_5 = \emptyset$ or $|\mathcal{W}_5| < 3$, then continue with the next a_5 (in view of Lemmas 6.10, 6.11, and 6.14).
    ii. For each $a_1 \in \mathcal{X}_1$:

(a) Compute $Y = F[W, a_5][W, a_1]$. If $|W/Y| > 2^3$, then continue with the next a_1 (in view of Corollary 6.12).
(b) Compute $\mathcal{R}_1 = \{r \in \mathcal{R}_5 \cap Y \mid r^{a_1} = r^{-1}\}$ and $\mathcal{W}_1 = \{w \in \mathcal{W}_5 \setminus Y \mid w^{a_1}w \notin T_1S_2 \cup S_3T_3\}$. If $\mathcal{R}_1 = \emptyset$ or $|\mathcal{W}_1| < 3$, then continue with the next a_1 (in view of Lemmas 6.10, 6.11, and 6.14).
(c) Replace $\mathcal{R}_1$ by its subset of all r such that conjugation by r on W coincides with $(a_5a_1)^2$. If $\mathcal{R}_1$ is empty, then continue with the next a_1 (in view of Lemma 6.10).
(d) Compute the set $\mathcal{W}_2$ of all members w of $\mathcal{W}_1$ such that $ww^{a_5a_1} \in \mathcal{R}_1$ and the girth test on $S = \{w\}$ is passed. If $|\mathcal{W}_2| < 3$, then continue with the next a_1 (in view of Lemma 6.14).
(e) For each $w_2 \in \mathcal{W}_2$:
   I. Compute $r = w_2w_2^{a_5a_1}$ and the set $\mathcal{W}_3$ of all $w \in \mathcal{W}_2 \setminus w_2Y$ such that $ww^{a_5a_1} = r$, $|w^{-1}w_2| > 2$, and the girth test on $S = \{w_2, w\}$ is passed. If $|\mathcal{W}_3| < 2$ or $r \notin \mathcal{R}_1$, then continue with the next w_2 (in view of Corollary 4.5 and Lemma 6.14).
   II. For each $w_3 \in \mathcal{W}_3$:
      -i- Compute the set $\mathcal{W}_4$ of all $w \in \mathcal{W}_3 \setminus w_3Y$ such that $|w^{-1}w_3| > 2$, and the girth test on $S = \{w, w_2, w_3\}$ is passed; if $|W/Y| = 8$, then also require $w \notin w_2w_3Y$.
      -ii- If $\mathcal{W}_4$ is empty, then continue with the next w_3 (in view of Corollary 4.5 or Lemma 6.14).
      -iii- For each $w_4 \in \mathcal{W}_4$, if $(a_5, a_1, w_2, w_3, w_4)$ passes the distance test, add it to the list $\mathcal{L}$.

Return $\mathcal{L}$.

The list $\mathcal{L}$ produced by applying the above algorithm to each tight factorization is empty.

Theorem 8.2. *None of the* 3090 *groups* W *of order 256 of Lemma 8.1 has a tight 42424-factorization* $W = T_1S_2T_2S_3T_3$ *that extends to a group* Q*, generated by involutions* $\alpha_1, \ldots, \alpha_5$*, with a tight 2424242-factorization* $Q = S_1T_1S_2T_2S_3T_3S_4$ *for which the conclusions of Lemmas 4.3, 6.2, 6.9, 6.10, 6.11, 6.14 and Corollaries 4.5, 5.2, 6.12, 6.13 with* $Q_0 = Q$ *are satisfied.*

Proof: If $\alpha_1, \ldots, \alpha_5$ were as stated in the assumptions, then $(a_5, a_1, w_2, w_3, w_4)$, where a_i (for $i = 1, 5$) is the automorphism induced on W by conjugation by α_i and $w_j = \alpha_5\alpha_j\alpha_1$ (for $j = 2, 3, 4$), would appear in the output $\mathcal{L}$ of the algorithm, contradicting that $\mathcal{L}$ is the empty list. □

This result implies Theorem 1.2 in case $Q_0 = Q$. Since the case $Q_0 < Q$ was resolved in Section 7, this completes the proof of Theorem 1.2.

We finish with some remarks on the computations. The steps in our algorithm are chosen so as to filter out cases as early as possible. In our GAP implementation, we performed all group operations in W (represented as a permutation group) without having to build Q; this was critical to the speed of the computations. The above interpretation of balanced words in α_i as elements of W is part of this effort. The number of tight factorisations for a group of order 256 varied from 1 to 60322. The most expensive calculations were those for 256#6823, 256#27519, and 256#27633, with 4456, 4101, and 27125 tight factorizations respectively. Again, the memory resources required to establish the result are small, but the time taken is significant: an estimate is approximately 15 CPU years running GAP 4.6.5.

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