The fixity of groups of prime-power order

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Abstract

It is well known that the only groups of prime-power order which can act fixed point freely on a complex linear space are the cyclic or generalised quaternion groups. Given a positive integer f and a prime p exceeding f, we determine the p-groups which have a faithful complex representation such that the dimensions of the spaces of fixed points of non-trivial elements are at most f.

1 Introduction

Let G be a group and let V be a finite-dimensional right $\mathbb{C}G$-module. We denote the fixed-point space of an element g of G by $C_V(g) = \{v \in V : vg = v\}$. The fixity, $fix(V)$, of V is the maximum of the dimensions of $C_V(g)$ over all non-trivial elements $g \in G$. We define the *fixity* of G to be the minimum of $fix(V)$ over all *faithful* $\mathbb{C}G$-modules V.

It is a well known result, which goes back to Burnside ([3], §248), that the p-groups of fixity zero are cyclic or generalised quaternion. Here, we determine the p-groups of fixity one for all primes p. More generally, given a positive integer f we classify the p-groups of fixity f for primes p greater than f.

The *p-rank* of a group G is the maximum of the ranks of elementary abelian p-subgroups of G. Our first result provides an abstract characterisation of p-groups with given fixity.

Theorem 1.1 *Let f be a non-negative integer and let p be a prime greater than f. A p-group G has fixity f if and only if the following conditions hold:*

(i) *G has p-rank $f+1$;*

(ii) *either G is abelian or G has cyclic centre and an abelian maximal subgroup.*

It follows immediately from Theorem 1.1 that the number of abelian groups of order p^n and fixity f is the number of partitions of n into $f+1$ parts. The number, $N(f, p^n)$, of non-abelian groups of order p^n and fixity f, where $p > f$, is computed below.

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Corollary 1.2 *If $n < f+2$, then $N(f,p^n)=0$.*

Proof. If G is a p-group of fixity f, then it has an elementary abelian subgroup of order p^{f+1}. If G is non-abelian, then $|G| \geq p^{f+2}$. □

Theorem 1.3 *Let f be a positive integer and let p be a prime.*

1. *For $p > f+2$,*

$$N(f,p^n) = \begin{cases} 2+\gcd(p-1,f+1) & n \geq f+3 \\ 2 & n = f+2. \end{cases}$$

2. *For $p = f+2$,*

$$N(f,p^n) = \begin{cases} f+5+\gcd(n-2,f+1) & n > f+3 \\ f+3 & n = f+3 \\ 3 & n = f+2 \geq 4 \\ 2 & n = 3 \text{ and } f = 1. \end{cases}$$

3. *For $p = f+1$,*

$$N(f,p^n) = \begin{cases} 2(n-f-2) + \sum_{k=f+2}^{n-2} \gcd(k-1,f) & n \geq f+3 \geq 5 \\ 1 & n = f+2 \geq 4 \\ 3n-8 & f = 1. \end{cases}$$

We see that the relationship between p and f is critical in determining the magnitude of $N(f,p^n)$. Indeed, if $p \geq f+2$, then $N(f,p^n)$ is bounded above in terms of f alone, while $N(f,p^n)$ tends to infinity with n when $p = f+1$. Of course, if neither $f+1$ nor $f+2$ is a prime, then only the first case of Theorem 1.3 applies.

Our count follows from a more general result, Theorem 4.2, which lists the p-groups of fixity f for all primes $p > f$.

The proof uses some results from Shalev [6] which studies the fixity of arbitrary finite groups.

2 Proof of Theorem 1.1

We need the following elementary results which follow from Shalev ([6], Lemmas 2.1 and 3.1).

Lemma 2.1 *A p-group of fixity f has p-rank at most $f+1$.*

Lemma 2.2 *A p-group G which has a $\mathbb{C}G$-module of fixity f and dimension greater than $(p+1)f$ is cyclic or generalised quaternion.*

This gives rise to the following two results.

Corollary 2.3 *The fixity of an abelian p-group of rank r is r − 1.*

Proof. Let G be an abelian group of rank r. By Lemma 2.1 the fixity of G is at least $r-1$. On the other hand G has a faithful module of dimension r, and so the fixity of G cannot exceed $r-1$. □

This proves Theorem 1.1 in the abelian case. We now consider non-abelian groups.

Corollary 2.4 *A non-abelian p-group of fixity f less than p has a faithful irreducible module of fixity f and dimension p.*

Proof. The result is clear if such a group, G, is generalised quaternion, so suppose otherwise. Then, since G is non-abelian, it has p-rank at least two.

Let V be a faithful $\mathbb{C}G$-module of fixity f. Since G is non-abelian and V is faithful and completely reducible, V has an irreducible submodule U of dimension p^k for some $k \geq 1$. The fixity of U is clearly at most f; since f is less than p, it follows that U is faithful and so has fixity at least f.

By Lemma 2.2,

$$dim(U) \leq (p+1)(p-1) = p^2 - 1.$$

Hence $k = 1$ and U is the required module. □

Irreducible p-subgroups of $GL(p, \mathbb{C})$ have been studied by Conlon [4]. His detailed description will be applied in Section 4. Here we only need the following elementary observation.

Lemma 2.5 *A non-abelian p-subgroup of* $GL(p, \mathbb{C})$ *is irreducible if and only if it has cyclic centre and an abelian maximal subgroup.*

The following is a straightforward consequence.

Corollary 2.6 *A non-abelian p-group of fixity less than p has cyclic centre and an abelian maximal subgroup.*

We now come to the main point of the discussion.

Theorem 2.7 *A p-group of fixity f less than p has p-rank f + 1.*

Proof. By Corollary 2.3 we may assume that such a group, G, is non-abelian. Let r denote the p-rank of G. Then, by Lemma 2.1, $r \leq f+1$. It remains to show that $r \geq f+1$.

By Corollary 2.4 there is a faithful irreducible $\mathbb{C}G$-module V of dimension p and fixity f. Let A be an abelian maximal subgroup of G and x an element of G which together with A generates G. As an A-module V has an irreducible one-dimensional submodule V_0 generated by v_0. For $i = 0, \ldots, p-2$ put $v_{i+1} = v_i x$; then $\{v_0, \ldots, v_{p-1}\}$ is a basis of V.

As in Corollary 2.4, it suffices to consider the case $r \geq 2$. So it remains to consider the case $f \geq 2$. Let a be an element of G whose space, $C_V(a)$, of fixed points in

V has dimension f. Since elements outside A are represented by monomial matrices corresponding to cycles of length p, their fixity is at most 1. It follows that $a \in A$. Since $C_V(a^p) \geq C_V(a)$, we can take a to have order p.

Let E be the elementary abelian subgroup generated by elements of order p in A, then E has rank r and $a \in E$. Let $\omega \in K$ be a primitive p^{th} root of unity and $\mathbb{F}_p$ be a field of p elements. Let W be the additive group of all polynomials of degree at most $p-1$ in $\mathbb{F}_p[t]$. Let $\phi : E \longrightarrow W$ be the mapping defined by $v_i e = \omega^{e\phi(i)} v_i$ for all e in E; here $e\phi(i)$ is the value of the associated polynomial function defined on the index set $\{0, \ldots, p-1\}$ identified with the field $\mathbb{F}_p$. Clearly ϕ is a monomorphism.

For $0 \leq k \leq p-1$ set $W_k = \{P \in W : P \text{ has degree at most } k\}$. Then clearly $\{0\} = W_{-1} \leq W_0 \leq W_1 \leq \ldots \leq W_{p-1} = W$ and $[W_k : W_{k-1}] = p$ for $0 \leq k \leq p-1$.

There is an automorphism σ of W defined by $P^\sigma(t) = P(t-1)$ for all P in W. Note that the W_k are σ-invariant subgroups of W and that $[W_k, \sigma] = W_{k-1}$ for $0 \leq k \leq p-1$. Hence σ acts uniserially on W and every σ-invariant subgroup of W has the form W_k for some k. Note that the rank of W_k is $k+1$.

Since $(e\phi)^\sigma = e^x\phi$ for all e in E, the image $E\phi$ is σ-invariant. It follows that $E\phi = W_k$ for some k. Since a has fixity f, the polynomial $a\phi$ has f roots and so degree at least f. Hence $E\phi = W_k$ for some $k \geq f$. Therefore $r \geq f+1$. □

We can now complete the proof of Theorem 1.1 by dealing with the non-abelian case. Theorem 2.7 and Corollary 2.6 show that a non-abelian p-group of fixity $f < p$ satisfies conditions (i) and (ii) of the theorem.

Conversely, let G be a non-abelian p-group satisfying conditions (i) and (ii). We have to show that the fixity of G is f. This will follow from Theorem 2.7 once we show that the fixity of G is less than p. It clearly suffices to show that G has a faithful module of dimension p. Since G has cyclic centre and an abelian maximal subgroup, this follows from Lemma 2.5. Theorem 1.1 is proved.

3 Non-abelian p-groups of fixity one

In Section 4, p-groups of fixity less than p are classified using Conlon's work. Clearly this provides a complete list of the p-groups of fixity one for all primes p.

In this section, we provide an alternative and perhaps simpler method of obtaining the non-abelian p-groups of fixity one, for odd p. Theorem 1.1 states that these groups have p-rank precisely 2. It follows from Theorem 4.1 of [2] that, for odd p and $n \geq 5$, a group of order p^n and p-rank 2 is either a metacyclic group, a 3-group of maximal class, or one of three exceptional types. We obtain the groups of fixity one among these by checking which of these groups have an abelian maximal subgroup and cyclic centre.

Newman & Xu [5] show that every metacyclic p-group, for odd p, has a presentation

$$\{\, a, b \;:\; a^{p^{r+s+u}} = 1,\; a^b = a^{1+p^r},\; b^{p^{r+s+t}} = a^{p^{r+s}} \,\}$$

where $r \geq 1$, $s, t, u \geq 0$, and $u \leq r$ and different values of the parameters r, s, t, u correspond to different isomorphism types. The metacyclic groups which have an abelian maximal subgroup and cyclic centre are those with $r = u = 1$ and $s = 0$. Note these groups also have split presentations $\{\, a, b \;:\; a^{p^{n-1}} = b^p = 1,\; a^b = a^{1+p^{n-2}} \,\}$.

Blackburn [1] determines the 3-groups of maximal class which have an abelian maximal subgroup; for all $n \geq 4$, there are four such groups of order 3^n when n is even and three when n is odd.

By examination, each of the three exceptional groups described by Blackburn has the desired property.

This, combined with an examination of the groups of order p^3 and p^4, yields the following result.

Theorem 3.1 *For odd p, the non-abelian p-groups of fixity one and order p^n are:*

1. *for $n \geq 3$, the metacyclic group*

$$\langle\, a, b \;:\; a^{p^{n-1}} = b^p = 1,\; a^b = a^{1+p^{n-2}} \,\rangle$$

and

$$\langle a, x, y : a^{p^{n-3}} = [x, y], [a, x] = [a, y] = x^p = y^p = 1\rangle;$$

2. *for $n \geq 4$, the 3-groups of maximal class*

$$\langle\, s, s_1, s_2, \ldots, s_{n-1} : [s_{i-1}, s] = s_i,\; s^3 = s_{n-1}^{\delta},\; s_1^3 s_2^3 s_3 = s_{n-1}^{\gamma},\; [s_i, s_j] = 1 \,\rangle$$

with $\gamma, \delta \in \{0, 1\}$ – where $\gamma\delta \neq 1$ if n is odd; and

$$\langle a, x : a^{p^{n-3}} = [x, a, x]^{\lambda}, x^p = 1, [x, a, a] = 1\rangle$$

where λ is either 1 or a non-quadratic residue.

In particular, for $p \geq 5$ and $n \geq 4$, there are four groups of fixity one and order p^n.

4 Non-abelian p-groups of fixity at most $p - 1$

Conlon [4] gives various descriptions for the non-abelian p-groups with cyclic centre and with an abelian maximal subgroup. In particular he gives presentations for such groups. We follow his naming of the groups and the generators but write the relations in a slightly different form. There is a non-abelian group P_{klm} of order p^{k+l} which has a generating set

$$x, y_1, \ldots, y_k, z_1, \ldots, z_l$$

and defining relations

$$z_1, \ldots, z_l \text{ are central},$$
$$z_{i+1}^p = z_i \text{ for } i = 1, \ldots, l-1 \;,$$
$$y_1, \ldots, y_k \text{ pairwise commute},$$
$$[y_{j+1}, x] = y_j \text{ for } j = 1, \ldots, k-1 \;,$$
$$y_1 = z_1 \;,$$
$$(y_j x)^p = 1 \text{ for } j = 1, \ldots, k-1 \;,$$
$$x^p = 1 \;,\; (y_k x)^p = z_l^m \text{ for } m = 0, \ldots, p-1 \;,$$
$$x^p = (y_k x)^p = z_l \text{ for } m = p$$

where $l \geq 1$, $k \geq 3$ and $0 \leq m \leq p$, or $k = 2$ and $m \in \{0, p\}$.

Theorem 4.1 *A non-abelian p-group with cyclic centre and an abelian maximal subgroup has p-rank at most p. Moreover P_{klm} has p-rank p when*

- $p = 2$, *unless* $l = 1$ *and* $m = 2$,
- $p \geq 3, l \geq 2, k > p$,
- $p \geq 3, l \geq 2, k = p, m \in \{0, p\}$,
- $p \geq 3, l = 1, k = p, m = 1$.

Otherwise P_{klm} has p-rank

- 1 *when* $p = 2$, $l = 1$, *and* $m = 2$ *(the generalised quaternion case),*
- 2 *when* $p \geq 3$ *and* $k = 2$,
- *the minimum of k and $p - 1$ when* $m \in \{0, p\}$,
- *the minimum of $k - 1$ and $p - 1$ when* $0 < m < p$.

Proof. Conlon observes that every such group can be embedded into the wreath product P of a p-quasicyclic group by a cyclic group of order p. The base group A has index p in P and is the direct product of p quasicyclic groups. Hence the maximal elementary abelian subgroups of P are

- $\Omega_1(A)$ the subgroup of A consisting of the elements of order dividing p – this has rank p, and
- the $\Omega_1(C_P(g))$ where g is an element of order p which does not lie in A – these have rank 2.

The first sentence of the theorem and the second sentence for $p = 2$ follow at once. So we can now assume $p \geq 3$. For all k the subgroup $Y_1 = \langle y_1, \ldots, y_h \rangle$ of P_{klm} is elementary abelian where h is the minimum of $k - 1$ and $p - 1$. For $l \geq 2, k > p$, the subgroup $\langle y_1, \ldots, y_{p-1}, y_p z_2 \rangle$ is elementary abelian. For $l \geq 2, k = p, m \in \{0, p\}$ and $l = 1, k = p, m = 1$, the subgroup $\langle y_1, \ldots, y_{p-1}, y_p \rangle$ is elementary abelian. This completes the proof of the first part of the second sentence. Further for $k < p, m \in \{0, p\}$ the subgroup $Y_2 = \langle y_1, \ldots, y_k \rangle$ is elementary abelian. When $l = 1, k > p$ and when $l = 1, k = p, m \neq 1$, every abelian normal subgroup of P_{k1m} properly containing Y_1 contains y_p – in the first case this can be seen by commuting an appropriate number of times with a suitable element of P_{k1m}. Since y_p has order p^2 in these cases, it follows that Y_1 is maximal elementary abelian. When $l \geq 2, k = p, m \neq 0$ or p and when $k < p, m \neq 0$ or p, every abelian normal subgroup properly containing Y_1 contains z_2 and it follows that Y_1 is maximal elementary abelian. Finally when $k < p, m \in \{0, p\}$, every abelian normal subgroup properly containing Y_2 contains z_2 and it follows that Y_2 is maximal elementary abelian. □

Given a positive integer f and a prime $p > f$, we now list the non-abelian groups of order p^n which have fixity f.

Theorem 4.2 *Let f be a positive integer and p a prime.*

1. *For $p > f+2$, the non-abelian groups of order p^n and fixity f are $P_{f+1,n-(f+1),m}$ where $m \in \{0,p\}$, and $P_{f+2,n-(f+2),m}$ where $0 < m < p$.*

2. *For $p = f+2$, the non-abelian groups of order p^n and fixity f are $P_{n-1,1,m}$ where $0 \leq m \leq p$, $P_{f+1,n-(f+1),m}$ where $m \in \{0,p\}$, and $P_{f+2,n-(f+2),m}$, where $0 < m < p$, except for $P_{f+2,1,1}$.*

3. *For $p = f+1$ and odd p, the non-abelian groups of order p^n and fixity f are $P_{k,n-k,m}$ where $f+2 \leq k \leq n-2$ and $0 \leq m \leq p$, $P_{p,n-p,m}$ where $m \in \{0,p\}$ and $P_{p,1,1}$.*

   The non-abelian 2-groups of fixity 1 are $P_{k,n-k,m}$ where $3 \leq k \leq n-2$ and $0 \leq m \leq 2$, $P_{2,n-2,m}$, where $m \in \{0,2\}$, and $P_{n-1,1,m}$ where $m \in \{0,1\}$,

Proof. This result follows from Theorems 1.1 and 4.1. □

Conlon ([4], Proposition 3.3) proves that the groups P_{kl0} and P_{klp} are not isomorphic and, for $k \geq 3$, neither is isomorphic to any of P_{klm} where $0 < m < p$. He also shows that the number of isomorphism types of groups P_{klm} where $0 \leq m \leq p$ is $2 + \gcd(k-1, p-1)$ when $k > 2$ and 2 when $k = 2$. (The case $l = 1$ follows from Theorem 4.3 of [1].)

Theorem 1.3 now follows from Theorem 4.2 and the above remarks. We leave the tedious verification to the reader.

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