On some questions about a family of cyclically presented groups

Alberto Cavicchioli, E.A. O'Brien and Fulvia Spaggiari

Abstract

We study various questions about the generalised Fibonacci groups, a family of cyclically presented groups, which includes as special cases the Fibonacci, Sieradski, and Gilbert-Howie groups.

1 Introduction

Consider the class of groups with *cyclic presentation*:

$$G_n(w) = \langle\, x_1, \ldots, x_n \,:\, w = 1,\, \theta(w) = 1, \ldots,\, \theta^{n-1}(w) = 1 \,\rangle$$

where w is a reduced word in the alphabet $X = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ and θ is the automorphism of the free group of rank n defined by setting $\theta(x_i) = x_{i+1} \bmod n$. One of the motivations for the study of these groups is their connection with the topology of closed connected orientable 3-manifolds; see, for example, [5, 12].

If $w = x_i x_{i+m} x_{i+k}^{-1}$, then we obtain the *generalised Fibonacci groups* introduced in [4]:

$$G_n(m, k) = \langle\, x_1, \ldots, x_n \,:\, x_i x_{i+m} = x_{i+k} \quad (i = 1, \ldots, n) \,\rangle$$

where the subscripts are taken modulo n.

For particular choices of parameters, these groups are well-known: $G_n(1, 2)$ are the *Fibonacci groups* $F(2, n)$ (see [7, 17]); $G_n(2, 1)$ are the *Sieradski groups* $S(n)$ (see [16, 18]); $G_n(m, 1)$ are the *Gilbert-Howie groups* $H(n, m)$ (see [9]).

We can immediately restrict our attention to those groups $G_n(m, k)$ whose parameters satisfy the conditions $0 < m < k < n$ and $(n, m, k) = 1$. Such groups are *irreducible*. Bardakov & Vesnin [2] prove:

This work was performed under the auspices of GNSAGA of CNR (National Research Council) of Italy. It was supported in part by the MIUR (Ministero dell'Istruzione, Università e Ricerca) of Italy within the project *Proprietà Geometriche delle Varietà Reali e Complesse*, and by the Marsden Fund of New Zealand via grant UOA 412. O'Brien thanks the Department of Mathematics at Modena for its hospitality while this work was completed. We thank the referee for many helpful suggestions, and Matthew Auger and Gerald Williams for their comments. 2000 *Mathematics Subject Classification.* Primary 20F05.

- if $G_n(m,k)$ is not irreducible, then it is either trivial, cyclic, or a free product of $G_{n'}(m',k')$ for smaller values of n',m',k';
- if $G_n(m,k)$ is irreducible and either $(n,k)=1$ or $(n,k-m)=1$, then $G_n(m,k)$ is isomorphic to $G_n(t,1)=H(n,t)$, where $tk \equiv m \bmod n$ or $t(k-m) \equiv (n-m) \bmod n$ respectively.

This motivates the following definition in [2]: $G_n(m,k)$ is *strongly irreducible* if it is irreducible and $(n,k)>1$ and $(n,k-m)>1$.

Bardakov & Vesnin [2] pose, and study, a number of questions about these groups. These include:

- Under what conditions is $G_n(m,k)$ aspherical? Finite and non-trivial?
- Determine the number of isomorphism types among $G_n(m,k)$.
- Determine the structure of the largest abelian quotient, $A_n(m,k)$, of $G_n(m,k)$.
- Under what conditions is $G_n(m,k)$ the fundamental group of a 3-orbifold (in particular, a hyperbolic closed 3-manifold) of finite volume?

We summarise recent progress in answering these questions.

With a few exceptions, Gilbert & Howie [9] identify those $H(n,m)$ which are aspherical or finite. Williams [19] proves that a strongly irreducible group $G_n(m,k)$ is not aspherical if and only if $(m,k)=1$ and either $n=2k$, or $n=2(k-m)$. He determines sufficient conditions for an irreducible group to be perfect. If, as he conjectures, these are also necessary, then every strongly irreducible group is not perfect; and he describes the structure of those which are finite and non-trivial. We show that $H(9,3)$ is infinite, thus reducing the undecided cases among irreducible (but not strongly irreducible) groups to 2.

Let $f(n)$ denote the number of isomorphism types among the irreducible groups $G_n(m,k)$. We obtain some new isomorphisms, and demonstrate that the known isomorphisms suffice to obtain $f(n)$ for all but four values of $n \leq 27$. We formulate a sharp conjecture for $f(p^\ell)$ where p is a prime.

Under the hypothesis of irreducibility, Corollary 5.8 of [5] shows that $A_n(m,k)$ is infinite if and only if $n \equiv 0 \bmod 6$, $m+k \equiv 3 \bmod 6$, and m is even. An equivalent result appears in [19, Theorem 4]. If $2k \equiv m \bmod n$, then we obtain a complete description of $A_n(m,k)$.

Corollary 3.5 of [5] is a slight improvement on [2, Theorem 3.1]: if n is odd and $(2k-m,n)=1$, then $G_n(m,k)$ cannot be the fundamental group of a hyperbolic closed 3-orbifold of finite volume. If $G_n(m,k)$ is irreducible and $2k \equiv m \bmod n$, then we show that $G_n(m,k) \cong S(n)$, the fundamental group of a closed connected orientable 3-manifold. Finally, we prove that the split extension of an irreducible $G_n(m,k)$ by a cyclic group of order n has a homomorphism onto a particular triangle group if both $(n,k)=1$ and $2(2k-m) \equiv 0 \bmod n$.

2 The isomorphism problem

The most general result on isomorphism is the following [2, Theorem 1.1].

Theorem 1. *Let $G_n(m,k)$ and $G_n(m',k')$ be irreducible groups. Assume that k' is divisible by $r=(n,k-m)$, $(n,k')=1$, and there exist integers $i\in\{1,\ldots,r\}$ and $j\in\{1,\ldots,n/r\}$ such that*

$$\begin{cases} i+j(k-m)\equiv(1-m) \bmod n \\ m'+1\equiv(i+jk') \bmod n. \end{cases}$$

Then $G_n(m,k)\cong G_n(m',k')$.

Observe that the extra condition, $(n,k')=1$, omitted from the original statement is both necessary and a consequence of the proof: for example, $\mathbb{Z}_7\cong G_6(1,3)\not\cong G_6(3,4)\cong\mathbb{Z}_2^3\rtimes\mathbb{Z}_7$.

Theorem 1 assumes both that k' is divisible by $(n,k-m)$ and $(n,k')=1$, so $r=1$. Hence, as was pointed out by the referee, we obtain an equivalent and simpler formulation.

Theorem 2. *Let $G_n(m,k)$ and $G_n(m',k')$ be irreducible groups and assume $(n,k')=1$. If $m'(m-k)\equiv mk' \bmod n$, then $G_n(m,k)$ is isomorphic to $G_n(m',k')$.*

We record some obvious consequences.

Corollary 3.

(1) *If $n\geq 5$ is odd, then $G_n(n-3,n-1)\cong G_n(n-3,n-2)$.*

(2) *$G_{2h+1}(h,h+1)\cong G_{2h+1}(h,2h)\cong G_{2h+1}(1,2)=F(2,2h+1)$.*

(3) *If $(2h+1,k-1)=1$, then $G_{2h+1}(1,k)\cong G_{2h+1}(1,2h+2-k)$.*

Proof. We illustrate the method by proving (3). By hypothesis, $(2h+1,k-1)=1$ and so $(2h+1,2h+2-k)=1$. Since $(1-k)\equiv(2h+2-k) \bmod (2h+1)$, the result follows. □

Corollary 4. *If there exists β such that $\beta s\equiv 1 \bmod n$ and $\beta(1-t)\equiv 1 \bmod n$, then $G_n(1,t)\cong G_n(1,s)$.*

Proof. Since $\beta s\equiv 1 \bmod n$, we conclude that $(n,s)=1$. □

Proposition 5. *If $(n,m)=1$, then $G_n(m,k)$ is isomorphic to $G_n(1,t)$, where $tm\equiv k \bmod n$.*

Proof. We rename the generators of $G_n(m,k)$: $c_1 = x_1, c_2 = x_{1+m}, \ldots, c_n = x_{1+(n-1)m}$. The first relation $x_1 x_{1+m} = x_{1+k}$ of $G_n(m,k)$ becomes $c_1 c_2 = c_{1+t}$, where $c_{1+t} = x_{1+tm} = x_{1+k}$ with $tm \equiv k \pmod{n}$. The next relation $c_2 c_3 = c_{2+t}$ corresponds to $x_{1+m} x_{1+2m} = x_{1+m+k}$ since $c_{2+t} = x_{1+(1+t)m} = x_{1+m+k}$. Similarly, $c_j c_{j+1} = c_{j+t}$ corresponds to $x_{1+(j-1)m} x_{1+jm} = x_{1+(j+t-1)m}$; that is, $x_{1+(j-1)m} x_{1+jm} = x_{1+(j-1)m+k}$. If j runs over $\{1, \ldots, n\}$, then $1+(j-1)m$, taken mod n, runs over the same set. Therefore $G_n(m,k) \cong G_n(1,t)$ where $tm \equiv k \bmod n$. □

Proposition 6.

(1) $G_n(m,k) \cong G_n(m, n+m-k) \cong G_n(n-m, n-m+k)$.

(2) *If* $(n,t) = 1$, *then* $G_n(m,k) \cong G_n(mt, kt)$.

(3) $G_{2h}(2h-1, 2h-2) \cong G_{2h}(2h-1, 1) \cong G_{2h}(1, 2h-1) \cong G_{2h}(1,2) = F(2,2h)$.

Proof.

(1) Taking the inverse relation of $x_i x_{i+m} = x_{i+k}$ and substituting i with $-i-m$, we get $x_{-i}^{-1} x_{-(i+m)}^{-1} = x_{-(i+m-k)}^{-1}$. Setting $y_i := x_{-i}^{-1}$ yields the relation $y_i y_{i+m} = y_{i+n+m-k}$ which defines $G_n(m, n+m-k)$. The second isomorphism, which appears as [2, Lemma 1.1], can be similarly established.

(2) Set $G = G_n(m,k) = \langle x_i : x_i x_{i+m} = x_{i+k} \rangle$ and $H = G_n(mt, kt) = \langle y_i : y_i y_{i+mt} = y_{i+kt} \rangle$. Let $\phi : G \to H$ be defined by setting $\phi(x_j) = y_{1+t(j-1)}$. The map ϕ is onto since $(n,t) = 1$. Furthermore, ϕ sends the defining relations of G to those of H:

$$\phi(x_i x_{i+m} x_{i+k}^{-1}) = y_{1+t(i-1)} y_{1+t(i+m-1)} y_{1+t(i+k-1)}^{-1} = y_j y_{j+mt} y_{j+kt}^{-1}$$

where $j = 1 + t(i-1)$. Thus ϕ is a homomorphism and, since it is invertible, it is an isomorphism.

(3) This follows from (1). □

We illustrate the previous results by identifying some isomorphisms among $G_{27}(m,k)$. Corollary 3 implies that $G_{27}(24,26) \cong G_{27}(24,25)$, $G_{27}(13,14) \cong G_{27}(13,26) \cong G_{27}(1,2) \cong F(2,27)$, and $G_{27}(1,18) \cong G_{27}(1,10)$. Corollary 4 implies that $G_{27}(1,11) \cong G_{27}(1,17)$. Proposition 5 implies that $G_{27}(2,5) \cong G_{27}(1,16)$. Proposition 6 implies that $G_{27}(2,5) \cong G_{27}(2,24) \cong G_{27}(25,3)$ and $G_{27}(2,5) \cong G_{27}(4,10) \cong G_{27}(8,20) \cong G_{27}(10,25) \cong G_{27}(14,8)$.

Proposition 7. *If* p *is an odd prime, then there are at most* $(p-1)/2$ *isomorphism types among the irreducible groups* $G_p(m,k)$.

Proof. If p is prime, then $(p, m) = 1$. Proposition 5 implies that $G_p(m, k) \cong G_p(1, t)$ for some $t \in \{2, \ldots, p-1\}$, where $tm \equiv k \bmod p$. Since $(p, t-1) = 1$, there exists β such that $\beta(1-t) \equiv 1 \bmod p$.

If $2 \leq t \leq (p+1)/2$, then $s = p+1-t$ satisfies $(p+1)/2 \leq s \leq p-1$. Corollary 4 now implies that $G_p(1, t) \cong G_p(1, s)$ since $\beta s = \beta(p+1-t) \equiv 1 \bmod p$. Hence the isomorphism types arise by choosing $t \in \{2, \ldots, (p+1)/2\}$, and so $f(p) \leq (p-1)/2$. $\square$

Our investigations, reported in Section 5, suggest the following stronger result.

Conjecture 8. *If $n = p^\ell$ for an odd prime p and positive integer ℓ, then $f(n) = p^\ell - \frac{(p-1)}{2}p^{(\ell-1)} - 1$. If $\ell > 2$, then $f(2^\ell) = 3(2^{\ell-2})$.*

3 The abelianisation of $G_n(m, k)$

We obtain a complete description of the abelianisation of $G_n(m, k)$ when $2k \equiv m \bmod n$, and so extend [19, Lemma 5].

Lemma 9. *Assume $2k \equiv m \bmod n$.*

- *$G_n(m, k)$ is perfect if and only if $n/(n, k) \equiv \pm 1 \bmod 6$.*
- *The abelianisation of $G_n(m, k)$ is isomorphic to $\mathbb{Z}^{2(n,k)}$, $\mathbb{Z}_3^{(n,k)}$, or $\mathbb{Z}_2^{2(n,k)}$ if and only if $n/(n, k) \equiv 0$, $n/(n, k) \equiv \pm 2$, or $n/(n, k) \equiv \pm 3 \bmod 6$ respectively.*

Proof. If $G_n(m, k)$ is irreducible, then $2k \equiv m \bmod n$ implies that $(n, k) = 1$. Proposition 6(2) implies that $G_n(m, k) \cong G_n(2k, k) \cong G_n(2, 1) = S(n)$. Recall from [6, Theorem 2.1] that $S(n) \cong \pi_1(M_n)$, where M_n is the n-fold cyclic cover of the 3-sphere, branched over the trefoil knot. Thus M_n is the Brieskorn manifold of [13] and its abelianisation is well-known – see, for example, [15, p. 304].

If $G_n(2k, k)$ is not irreducible, then [2, Lemma 1.2] shows that $G_n(2k, k)$ is isomorphic to a free product of (n, k) copies of $G_{n/(n,k)}(2k/(n, k), k/(n, k))$, which is irreducible. Hence the abelianisation of $G_n(2k, k)$ is trivial, $\mathbb{Z}^{2(n,k)}$, $\mathbb{Z}_3^{(n,k)}$, or $\mathbb{Z}_2^{2(n,k)}$ according to the stated congruence conditions. $\square$

In summary, Proposition 6(2), [6, Corollary 2.2] and [18, Theorems B-C] imply the following: if $G_n(m, k)$ is irreducible and $2k \equiv m \bmod n$, then $G_n(m, k)$ is infinite if and only if $n \geq 6$, and it has a free subgroup of rank 2 when $n \geq 7$.

We now briefly discuss $A_n(1, t)$ for arbitrary n. It is well-known that $A_n(1, 2)$ is finite, of order $L_n - 1 - (-1)^n$, where L_n is the n-th Lucas number (see, for example, [11, Chapter 6]).

Proposition 10. *The structure of $A_n(1,t)$, where $t \in \{2, \dots, n-1\}$, is determined by the diagonal form of the integral $t \times t$ matrix*

$$\begin{pmatrix} a_{n+1}-1 & a_{n+t} & a_{n+t-1} & \dots & a_{n+3} & a_{n+2} \\ a_{n+2} & a_{n+t+1}-1 & a_{n+t} & \dots & a_{n+4} & a_{n+3} \\ a_{n+3} & a_{n+t+2} & a_{n+t+1}-1 & \dots & a_{n+5} & a_{n+4} \\ \vdots & \dots & \dots & \dots & \vdots & \vdots \\ a_{n+t-1} & a_{n+2t-2} & a_{n+2t-3} & \dots & a_{n+t+1}-1 & a_{n+t} \\ a_n+1 & a_{n+t-1} & a_{n+t-2} & \dots & a_{n+2} & a_{n+1}-1 \end{pmatrix}$$

where $a_i + a_{i+1} = a_{i+t}$ $(i \geq 1)$ and $a_1 = 1,\ a_i = 0$ $(2 \leq i \leq t)$.

Proof. We sketch a proof. The generators, $x_1, \dots, x_n$, of $A_n(1,t)$ commute and satisfy the following relations:

$$\begin{aligned} x_1x_2 &= x_{t+1} \\ x_2x_3 &= x_{t+2} \\ x_3x_4 &= x_{t+3} \\ &\vdots \\ x_{t-1}x_t &= x_{2t-1} \\ x_tx_{t+1} &= x_{2t} \\ x_{t+1}x_{t+2} &= x_{2t+1} \\ x_{t+2}x_{t+3} &= x_{2t+2} \\ &\vdots \\ x_{n-t}x_{n-t+1} &= x_n \end{aligned}$$

Hence $\{x_1, \dots, x_t\}$ generate $A_n(1,t)$, and $x_i = x_1^{a_i}x_2^{b_i^2}\cdots x_t^{b_i^t}$ for $i > t$.

We use the relations $x_i = x_{i-t}x_{i-t+1}$ and those implied by commutativity to deduce that

$$\begin{aligned} a_i &= a_{i-t} + a_{i-t+1} & i > t \\ b_i^j &= b_{i-t}^j + b_{i-t+1}^j & 2 \leq j \leq t \end{aligned}$$

where $a_1 = 1, a_i = 0, b_i^j = \delta_{i,j}$ for $2 \leq i,j \leq t$. Thus $b_i^j = a_{i+t-j+1}$ for all $i \geq 1$.

Hence the structure of $A_n(1,t)$ can be deduced from the diagonal form of the $t \times t$ matrix:

$$\begin{pmatrix} a_{n-t+1}+a_{n-t+2}-1 & b_{n-t+1}^2+b_{n-t+2}^2 & b_{n-t+1}^3+b_{n-t+2}^3 & \cdots & b_{n-t+1}^t+b_{n-t+2}^t \\ a_{n-t+2}+a_{n-t+3} & b_{n-t+2}^2+b_{n-t+3}^2-1 & b_{n-t+2}^3+b_{n-t+3}^3 & \cdots & b_{n-t+2}^t+b_{n-t+3}^t \\ a_{n-t+3}+a_{n-t+4} & b_{n-t+3}^2+b_{n-t+4}^2 & b_{n-t+3}^3+b_{n-t+4}^3-1 & \cdots & b_{n-t+3}^t+b_{n-t+4}^t \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1}+a_n & b_{n-1}^2+b_n^2 & b_{n-1}^3+b_n^3 & \cdots & b_{n-1}^t+b_n^t \\ a_n+1 & b_n^2 & b_n^3 & \cdots & b_n^t-1 \end{pmatrix}$$

The result now follows readily. $\square$

Of course, the isomorphism type of $G_n(1,t)$ is not determined by its abelian quotient invariants: $A_{11}(1,3) \cong A_{11}(1,4) \cong \mathbb{Z}_{23}$ but $G_{11}(1,3) \not\cong G_{11}(1,4)$ (since their derived groups have abelian quotient invariants 2^{11} and 3^{11} respectively).

4 Split extensions

Let $E_n(m,k)$ denote the split extension of $G_n(m,k)$ by $\mathbb{Z}_n = \langle \theta : \theta^n = 1 \rangle$, where θ is the automorphism sending each generator x_i to x_{i+1} (subscripts taken modulo n). The relations $x_i x_{i+m} = x_{i+k}$ of $G_n(m,k)$ imply

$$x\theta^{-m}x\theta^m = \theta^{-k}x\theta^k$$

where $x := x_n$, and $x_i = \theta^{-i}x\theta^i$. Setting $y = \theta^m x^{-1}$ (and eliminating $x = y^{-1}\theta^m$) yields

$$E_n(m,k) = \langle \theta, y : \theta^n = 1,\ \theta^{k-m}y^2 = y\theta^k \rangle.$$

Assume $2k \equiv m \bmod n$. As we observed in Lemma 9, if $G_n(m,k)$ is irreducible, then it is isomorphic to $G_n(2,1) = S(n)$. Further $E_n(m,k)$ is isomorphic to the fundamental group of the 3-dimensional orbifold whose underlying space is the 3-sphere and whose singular set is the trefoil knot with branching index n (see for example [6, Theorem 2.1]).

Lemma 11. *If $n \geq 3$ is odd, then $G_n(1,(n+1)/2) \cong S(n)$ is isomorphic to the derived group of the centrally extended triangle group*

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \ :\ \gamma_1^n = \gamma_2^2 = \gamma_3^3 = \gamma_1\gamma_2\gamma_3 \rangle.$$

If $n \geq 7$ is odd, then the centre of $G_n(1,(n+1)/2)$ is $\mathbb{Z}$, otherwise it is $\mathbb{Z}_2$. If $p \geq 5$ is a prime, there is a homomorphism from $G_p(1,(p+1)/2)$ onto $\mathrm{SL}(2,p)$. Furthermore, $G_5(1,3) \cong \mathrm{SL}(2,5)$ and $G_3(1,2) \cong Q_8$.

Proof. The first two assertions follow from [13, Section 3] since $G_n(1,(n+1)/2)$ is isomorphic to the fundamental group of the Brieskorn manifold $M(n,2,3)$. To prove the third, we define a map from $E_p(1,(p+1)/2) \to \mathrm{SL}(2,p)$:

$$\theta \to A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad\qquad y \to B = \begin{pmatrix} 0 & -2 \\ (p+1)/2 & 1 \end{pmatrix}.$$

One can easily verify that this is an epimorphism which induces an epimorphism from $G_p(1,(p+1)/2)$ to $\mathrm{SL}(2,p)$, sending $x_i \longmapsto A^iB^{-1}A^{i+1}$. The last follows from [9]. □

Theorem 12. *Let $G_n(m,k)$ be irreducible.*

(a) *If $2(2k-m) \equiv 0$ mod n, then $E_n(m,k)$ has a homomorphism onto the subgroup of $\mathrm{SL}(2,\mathbb{C})$ having presentation*

$$\{A, B : A^n = B^3 = 1, A^{2k-m} = (BA^k)^2\}.$$

(b) If $(n,k)=1$, then $E_n(m,k)$ has a homomorphism onto the group defined by the presentation $\{u,v : v^n=1, (uv)^3=1, v^{-\eta(2k-m)}=u^2\}$ where $\eta k \equiv 1 \bmod n$, for some integer η.

(c) If $(n,k)=1$ and $2(2k-m)\equiv 0$ mod n, then $E_n(m,k)$ covers the triangle group of type $(n,2,3)$ if n is odd, and of type $((n,2k-m),2,3)$ if n is even. If $(n,k)=1$, $2(2k-m)\equiv 0$ mod n and $(n,2k-m)\geq 6$, then $G_n(m,k)$ is infinite.

Proof. Recall $E_n(m,k)=\langle \theta, y : \theta^n=1, \theta^{k-m}y^2=y\theta^k\rangle$.

(a) We will exhibit a homomorphism $E_n(m,k)\to \mathrm{SL}(2,\mathbb{C})$ which both satisfies the relations of $E_n(m,k)$ and sends

$$\theta \to A=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \qquad\qquad y\to B=\begin{pmatrix}\alpha & \beta\\ 1 & \gamma\end{pmatrix}$$

where $\lambda^n=1$, $\beta\neq 0$, and $\alpha\gamma-\beta=1$. Such a homomorphism implies that

$$\begin{pmatrix}\alpha & \beta\\ 1 & \gamma\end{pmatrix}^2=\begin{pmatrix}\lambda^{m-k} & 0\\ 0 & \lambda^{k-m}\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 1 & \gamma\end{pmatrix}\begin{pmatrix}\lambda^{k} & 0\\ 0 & \lambda^{-k}\end{pmatrix}.$$

This gives the system of equations

$$\begin{cases}\alpha^2-\alpha\lambda^m+\beta=0\\ \beta(\alpha+\gamma)=\beta\lambda^{m-2k}\\ \alpha+\gamma=\lambda^{2k-m}\\ \gamma^2-\gamma\lambda^{-m}+\beta=0\\ \beta=\alpha\gamma-1.\end{cases}$$

Since $\beta\neq 0$, the second and third equations imply $\lambda^{2(2k-m)}=1$, which holds because n divides $2(2k-m)$. The system has the unique solution given by

$$\alpha=\frac{1}{\lambda^{m-2k}-\lambda^m} \qquad\qquad \beta=\frac{\lambda^{2(m-k)}-\lambda^{2m}-1}{(\lambda^{m-2k}-\lambda^m)^2} \qquad\qquad \gamma=\frac{-\lambda^{2(m-k)}}{\lambda^{m-2k}-\lambda^m}.$$

Assume $\lambda^m(\lambda^{-2k}-1)=0$. Then $|\lambda^m||\lambda^{-2k}-1|=0$, and so $2k\equiv 0 \pmod n$. But $G_n(m,k)$ is irreducible and so $0<m<k<n$. Since $2(2k-m)\equiv 0$ mod n, we deduce that $2m\equiv 0$ mod n, a contradiction. Hence $\lambda^m(\lambda^{-2k}-1)\neq 0$.

Let $\tau(B)$ be the square of the trace of the matrix B. Then

$$\tau(B)=\frac{(1-\lambda^{2(m-k)})^2}{(\lambda^{m-2k}-\lambda^m)^2}=\frac{1+\lambda^{4(m-k)}-2\lambda^{2(m-k)}}{1+\lambda^{2m}-2\lambda^{2(m-k)}}=1$$

since $\lambda^{2(m-2k)} = 1$ and $\lambda^{4(m-k)} = \lambda^{2m}$ as $2(2k-m) \equiv 0 \bmod n$. Hence B is elliptic. By [1, p. 39], we determine the multiplier M^2 of B by applying the quadratic formula

$$M^2 = \frac{1}{2}[\tau(B) - 2 \pm \sqrt{-4\tau(B) + \tau^2(B)}]$$

Since $M^2 = (-1 \pm i\sqrt{3})/2$, we conclude that B has order 3. The statement follows.

(b) If $y^3 = 1$, then the relation $\theta^{k-m}y^2 = y\theta^k$ becomes $\theta^{k-m} = y\theta^k y$, hence $\theta^{2k-m} = (y\theta^k)^2$. Thus adding the relation $y^3 = 1$ gives a homomorphism from $E_n(m,k)$ onto $\langle \theta, y : \theta^n = y^3 = 1, \theta^{2k-m} = (y\theta^k)^2 \rangle$. If $(n,k) = 1$, then there exist integers ξ and η such that $\xi n + \eta k = 1$. Setting $u = y\theta^k$ and $v = \theta^{-k}$, we deduce that $E_n(m,k)$ covers the group defined in (b).

(c) If n is odd, then by (b) $E_n(m,k)$ covers $\langle v, u : v^n = u^2 = (uv)^3 = 1 \rangle$. If n is even, then the relation $v^{(n,2k-m)} = 1$ implies a homomorphism of $E_n(m,k)$ onto the triangle group of type $((n, 2k-m), 2, 3)$. The infiniteness claim now follows from [8, §6.4]. $\square$

Consider the case when $(n,k) = 1$ and $2(2k-m) \equiv 0 \bmod n$. If n is also odd, then $2k \equiv m \bmod n$; since $G_n(m,k) \cong S(n)$, it is infinite for $n \geq 6$. If n is even, then (c) has new consequences: for example, it implies that $G_{12}(4,5)$ is infinite.

5 Investigating $G_n(m,k)$ for small values of n

We investigated the irreducible groups $G_n(m,k)$ for values of $n \leq 27$. We used implementations in Magma [3] of algorithms to perform coset enumerations, compute abelian quotient invariants and (normal) subgroups of low index, and construct presentations for subgroups and p-quotients of finitely-presented groups. We refer the interested reader to [10, Chapters 5 and 9] for details and references to these algorithms.

5.1 Isomorphism

We sought to solve the isomorphism problem among the irreducible $G_n(m,k)$ for small values of n. We applied the isomorphisms identified in Theorem 2, its corollaries, and Propositions 5-6 to obtain both an upper bound $U(n)$ to the value of $f(n)$, and a potentially redundant list of isomorphism types. We then used invariants of groups in the resulting list to obtain a lower bound $L(n)$ to the value of $f(n)$. These bounds frequently coincided, so allowing us to deduce the precise value of $f(n)$.

In most cases, it sufficed to compute the abelian quotient invariants of a group and those of its derived group to distinguish it from any other on the list. We note the exceptional cases.

- We proved that $G_{14}(1,3)$ is not isomorphic to $G_{14}(1,5)$ by showing that, among their normal subgroups of index 16, the number of distinct abelian quotient invariants is 8 and 9 respectively.
- The p-class 2 241-quotient of the derived group of $G_{22}(1,5)$ has order 241^{22}; the corresponding quotient of the derived group of $G_{22}(1,7)$ has order 241^{44}.
- $\mathrm{PSL}(2,5)$ is a homomorphic image of $G_{25}(1,3)$ but not of $G_{25}(1,6)$.
- $G_{26}(1,13)$ is finite, $G_{26}(13,14)$ is infinite.

We summarise our results in Table 1. For $n \in \{3,\ldots,27\}$, we record the values of $L(n)$ and $U(n)$; for each of the $U(n)$ groups, we list one defining value of the parameters (m,k). For $n \in \{17,19,21,23\}$, the values of $L(n)$ and $U(n)$ differ by 1. The unresolved cases are listed in Table 2.

Table 1 demonstrates that Conjecture 8 is sharp. For $n \in \{28,\ldots,200\}$, we computed $U(n)$ and counted the number of distinct abelian quotient invariants among $G_n(m,k)$. This provided additional evidence for the correctness of Conjecture 8; it also suggests that there is at most one coincidence among the values of the abelian quotient invariants of $G_n(m,k)$ when $n = p^\ell$.

5.2 Finiteness

We summarise the results of Gilbert & Howie [9] and Williams [19], with known isomorphisms applied.

Theorem 13.

(i) Suppose $(n,m) \notin \{(8,3),(9,3),(9,4),(9,7)\}$*. Then* $H(n,m)$ *is finite if and only if* $m = 0$ *or* 1*, or* $(n,m) = (2\ell, \ell+1)$ *where* $\ell \geq 1$*, or*

$$(n,m) \in \{(3,2),(4,2),(5,2),(5,3),(6,3),(7,4)\}.$$

(ii) Let $G = G_n(m,k)$ *be strongly irreducible and assume* $G \neq 1$*. Then* G *is finite if and only if* $(m,k) = 1$ *and* $n = 2k$ *or* $n = 2(k-m)$*, in which case* $G \cong \mathbb{Z}_s$ *where* $s = 2^{n/2} - (-1)^{m+(n/2)}$*.*

The structure of (the then known) finite irreducible groups among $H(n,m)$ is recorded in [9, Table 1]. Some of the exceptions from Theorem 13(i) have since been resolved. We now know that $H(8,3) \cong G_8(3,1)$ is a soluble group of order $3^{10} \cdot 5$ and derived length 3. First established by R.M. Thomas, its order can now be determined by a routine coset enumeration in Magma.

We now prove that $H(9,3)$ is infinite. Recall first Newman's extension [14] of the Golod-Šafarevič theorem, which we summarise for the prime 2.

n	$L(n)$	$U(n)$	Parameters (m,k)
3	1	1	$(1,2)$
4	2	2	$(1,2),\ (2,3)$
5	2	2	$(1,k)\ \ k \in \{2,\ 3\}$
6	5	5	$(1,k)\ \ k \in \{2,\ 3\},\ (2,3),\ (3,4),\ (4,5)$
7	3	3	$(1,k)\ \ k \in \{2,3,4\}$
8	6	6	$(1,k)\ \ k \in \{2,3,4\},\ (2,3),\ (2,5),\ (4,5)$
9	5	5	$(1,k)\ \ k \in \{2,\ldots,5\},\ (3,4)$
10	8	8	$(1,k)\ \ k \in \{2,\ldots,5\},\ (2,k)\ \ k \in \{3,5\},\ (4,7),\ (5,6)$
11	5	5	$(1,k)\ \ k \in \{2,\ldots,6\}$
12	12	12	$(1,k)\ \ k \in \{2,\ldots,6\},\ (2,k)\ \ k \in \{3,7\},\ (3,k)\ \ k \in \{4,5\}$ $(4,k)\ \ k \in \{5,7\},\ (6,7)$
13	6	6	$(1,k)\ \ k \in \{2,\ldots,7\}$
14	11	11	$(1,k)\ \ k \in \{2,\ldots,7\},\ (2,k)\ \ k \in \{3,5,7\},\ (4,9),\ (7,8)$
15	12	12	$(1,k)\ \ k \in \{2,\ldots,8\},\ (3,k)\ \ k \in \{4,5,7\},\ (5,6),\ (5,7)$
16	12	12	$(1,k)\ \ k \in \{2,\ldots,8\},\ (2,k)\ \ k \in \{3,5,9\},\ (4,5),\ (8,9)$
17	7	8	$(1,k)\ \ k \in \{2,\ldots,9\}$
18	17	17	$(1,k)\ \ k \in \{2,\ldots,9\},\ (2,k)\ \ k \in \{3,5,7,9\},\ (3,k)\ \ k \in \{4,7\}$ $(4,11),\ (6,7),\ (9,10)$
19	8	9	$(1,k)\ \ k \in \{2,\ldots,10\}$
20	18	18	$(1,k)\ \ k \in \{2,\ldots,10\},\ (2,k)\ \ k \in \{3,5,11\}$ $(4,k)\ \ k \in \{5,7,11\},\ (5,6),\ (5,8),\ (10,11)$
21	15	16	$(1,k)\ \ k \in \{2,\ldots,11\},\ (3,k)\ \ k \in \{4,5,7,8\},\ (7,8),\ (7,9)$
22	17	17	$(1,k)\ \ k \in \{2,\ldots,11\},\ (2,k)\ \ k \in \{3,5,7,9,11\},\ (4,13),\ (11,12)$
23	10	11	$(1,k)\ \ k \in \{2,\ldots,12\}$
24	26	26	$(1,k)\ \ k \in \{2,\ldots,12\},\ (2,k)\ \ k \in \{3,5,7,13\},\ (3,k)\ \ k \in \{4,5,8,10\}$ $(4,k)\ \ k \in \{5,7\},\ (6,k)\ \ k \in \{7,13\},\ (8,k)\ \ k \in \{9,13\},\ (12,13)$
25	14	14	$(1,k)\ \ k \in \{2,\ldots,13\},\ (5,6),\ (5,7)$
26	20	20	$(1,k)\ \ k \in \{2,\ldots,13\},\ (2,k)\ \ k \in \{3,5,7,9,11,13\},\ (4,15),\ (13,14)$
27	17	17	$(1,k)\ \ k \in \{2,\ldots,14\},\ (3,k)\ \ k \in \{4,5,10\},\ (9,10)$

Table 1: Lower and upper bounds for $f(n)$ for $n \leq 27$

n	Parameters (m, k)
17	$(1, 3),\ (1, 4)$
19	$(1, 3),\ (1, 6)$
21	$(1, 6),\ (1, 9)$
23	$(1, 3),\ (1, 7)$

Table 2: Possible isomorphisms

Theorem 14. *Let G be a group with a finite presentation on b generators and r relations. Let $G_1 := [G, G]G^2$ and $G_2 := [G_1, G]G_1^2$, where the elementary abelian 2-groups G/G_1 and G_1/G_2 have rank d and e respectively. If $r - b \leq d^2/2 + d/2 - d - e + (e - d/2 - d^2/4)d/2$, then G is infinite.*

Lemma 15. *The group $H := H(9, 3) \cong G_9(3, 4)$ is infinite.*

Proof. The second derived group, K, of H has index 448 in H. We obtain, using a Reidemeister-Schreier rewriting procedure [10, §2.5], a presentation for K on 321 generators and 768 relations. Now K has abelian quotient invariants $2^{36}4^7$. Let Q denote its 2-quotient of p-class 2: Q has order 2^{604}, its Frattini quotient has rank $d = 43$, and so $e = 561$. Theorem 14 implies that K is infinite. $\square$

The other exceptions, $H(9, 4) \cong G_9(1, 3)$ and $H(9, 7) \cong G_9(1, 4)$, remain unresolved.

References

[1] James W. Anderson. Hyperbolic Geometry. Springer Undergraduate Mathematics Series. Springer-Verlag, London 1999.

[2] V.G. Bardakov and A.Yu. Vesnin. A generalization of Fibonacci groups. *Algebra and Logic* **42** (2), 2003, 73–91.

[3] W. Bosma, J. Cannon, and C. Playoust. The MAGMA algebra system I: The user language. *J. Symbolic Comput.* **24** (1997), 235–265.

[4] Alberto Cavicchioli, Friedrich Hegenbarth, and Dušan Repovš. On manifold spines and cyclic presentations of groups. *Knot Theory* (Warsaw), Banach Center Publ. **42**, 1998, 49–55.

[5] Alberto Cavicchioli, Dušan Repovš, and Fulvia Spaggiari. Topological properties of cyclically presented groups. *J. Knot Theory Ramifications* **12** (2003), 243–268.

[6] A. Cavicchioli, F. Hegenbarth, and A.C. Kim. A geometric study of Sieradski groups. *Algebra Colloq.* **5** (2), 1998, 203–217.

[7] J.H. Conway. Advanced problem 5327. *Amer. Math. Monthly* **72**, 1965, 915.

[8] H.S.M. Coxeter and W.O.J. Moser. *Generators and Relations for Discrete Groups*. 4th ed. Springer-Verlag (Berlin), 1980, ix+169 pp.

[9] N.D. Gilbert and James Howie. LOG groups and cyclically presented groups. *J. Algebra*, **174**, 1995, 118–131.

[10] Derek F. Holt, Bettina Eick, and Eamonn A. O'Brien. *Handbook of computational group theory.* Chapman and Hall/CRC, London, 2005.

[11] D.L. Johnson. *Presentations of Groups.* London Math. Soc. Student Texts **15**. Cambridge Univ. Press, Cambridge, 1990.

[12] C. Maclachlan. Generalizations of Fibonacci numbers, groups and manifolds. Combinatorial and geometric group theory, Edinburgh 1993. London Math. Soc. Lecture Note Ser., **204**, 1995, 233–238.

[13] John Milnor. On the 3-dimensional Brieskorn manifolds $M(p,q,r)$. In: Knots, groups and 3-manifolds, Ann. of Math. Studies, vol. **84**, Princeton Univ. Press, Princeton, 1975, pp. 175-225.

[14] M.F. Newman. Proving a group infinite. *Arch. Math.* (Basel) **54** (1990), no. 3, 209–211.

[15] Dale Rolfsen. *Knots and Links.* Mathematics Lecture Series **7**, Publish or Perish Inc. Berkeley, 1976.

[16] Allan J. Sieradski. Combinatorial squashings, 3-manifolds, and the third homology of groups. *Invent. Math.* **84**, 1986, 121–139.

[17] Richard M. Thomas. The Fibonacci groups revisited. Groups—St. Andrews 1989, Vol. 2, 445–454, London Math. Soc. Lecture Note Ser., 160, Cambridge Univ. Press, Cambridge, 1991.

[18] Richard M. Thomas. On a question of Kim concerning certain group presentations. *Bull. Korean Math. Soc.* **28** (1991), no. 2, 219–224.

[19] Gerald Williams. The aspherical Cavicchioli–Hegenbarth–Repovš Generalized Fibonacci groups. To appear *J. Group Theory*, 2008.

Alberto Cavicchioli and Fulvia Spaggiari, Dipartimento di Matematica, Università di Modena e Reggio E., Via Campi 213/B, 41100 Modena, Italy

E-mail: cavicchioli.alberto@unimo.it spaggiari.fulvia@unimo.it

E.A. O'Brien, Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand

E-mail: e.obrien@auckland.ac.nz