The Alperin and Dade conjectures for the Fischer simple group Fi_{23}

Jianbei An E.A. O'Brien

Abstract

We modify the local strategy of [2] and use it to classify the radical subgroups and chains of the Fischer simple group Fi_{23}. We verify the Alperin weight conjecture and the Dade final conjecture for this group.

1 Introduction

In [2] we presented a local strategy to decide the Alperin and Dade conjectures for the finite simple groups and demonstrated its computational effectiveness by using it to verify these conjectures for the Conway simple group Co_2. In this paper, we develop a modification of this strategy and use it to verify the Alperin and Dade conjectures for the Fischer simple group Fi_{23}. Although the outlines of our computations and proofs are similar to those for Co_2, the details are significantly more complex.

We face two central challenges in attempting to decide these conjectures for Fi_{23}. The first is to determine the radical subgroups of Fi_{23} and hence to obtain its radical chains. In practice, some of its radical chains cannot be determined explicitly using existing approaches. Our *local strategy* and its modification presented here use knowledge of both the maximal and p-local subgroup structure of Fi_{23} to determine its radical subgroups. Second, we must determine the character tables of the normalizers of radical 2- and 3-chains of Fi_{23}. The character tables of the normalisers of some of the radical chains could not be calculated directly from the given representation using either of GAP [12] or MAGMA [3]. If the relevant normalizer is a maximal subgroup of a finite simple group, then its character table is stored in a library supplied with GAP. Otherwise, in some cases, we constructed a "useful" representation of the normalizer and attempted to compute directly its character table; if this construction failed, we used lifting of characters of quotient groups, induction and decomposition of characters of subgroups of the normalizer to obtain its character table. We outline the details in Section 6.

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Dade [7] generalized the Knörr-Robinson version of the Alperin weight conjecture

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and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p-blocks of some p-local subgroups of G. Dade [8] announced that his final conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has both trivial Schur multiplier and outer automorphism group, then the ordinary conjecture is equivalent to the final conjecture. We verify the Alperin weight conjecture and the Dade ordinary conjecture, and so the final one, for Fi_{23}.

The paper is organized as follows. In Section 2, we fix notation and state the two conjectures in detail. In Section 3, we develop our modified local strategy and explain how we applied it to determine the radical subgroups of Fi_{23}. In Section 4, we classify the radical subgroups of Fi_{23} up to conjugacy and verify the Alperin weight conjecture. In Section 5, we do some cancellations in the alternating sum of Dade's conjecture when $p = 2$ or 3, and then determine radical chains (up to conjugacy) and their local structures. In the last section, we verify Dade's conjecture. Three appendices provide details of proofs and character tables.

2 The Alperin and Dade Conjectures

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the maximal normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a *B-weight* if φ has p-defect 0 (see (5.5) of [7] for the definition) and $B(\varphi)^G = B$ (in the sense of Brauer), where $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{2.1}$$

of a finite group G, define $|C| = n$, $C_k : P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n). \tag{2.2}$$

The chain C is said to be *radical* if it satisfies the following two conditions: (a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for $1 \leq k \leq n$.
Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. For $B \in \mathrm{Blk}(G)$ and integer $d \geq 0$, let $\mathrm{k}(N(C), B, d)$ be the number of characters in the set

$$\mathrm{Irr}(N(C), B, d) = \{\psi \in \mathrm{Irr}(N(C)) : B(\psi)^G = B,\ \mathrm{d}(\psi) = d\},$$

where $\mathrm{d}(\psi)$ is the defect of ψ.

Dade's Ordinary Conjecture [7]. If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, then for an integer $d \geq 0$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = 0, \tag{2.3}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

3 A modification of the local strategy

The maximal subgroups of Fi_{23} were classified by Flaass [9] and Wilson [14]. Using this classification, we know that each radical 2- or 3-subgroup R of Fi_{23} is radical in one of the maximal subgroups M of Fi_{23} and further that $N_{\mathrm{Fi}_{23}}(R) = N_M(R)$.

A modified version of the local strategy of [2] was developed to classify the radical subgroups R. We review this method here and develop the necessary modification.

Step (1). We first consider the case where M is a p-local subgroup. Let $Q = O_p(M)$, so that $Q \leq R$. We find all the subgroup classes of a Sylow 2-subgroup D of M containing Q. Using MAGMA, we explicitly compute the quotient M/Q and the natural homomorphism $\eta : M \longrightarrow M/Q$. This approach provides a regular representation for M/Q, whose (potentially large) degree is usually computationally limiting. Hence, we construct a power-conjugate presentation for the quotient group $\eta(D) = D/Q$ since such presentations are computationally very effective. We now compute all subgroup classes in D/Q. The preimages in D of the subgroup classes of D/Q are the subgroup classes of D containing Q.

Available computational resources limit our ability to apply this approach directly to some maximal p-local subgroups of Fi_{23}. For example, when $p = 2$ and $M = 2.\mathrm{Fi}_{22}$, we could not construct the natural homomorphism $\eta : M \longrightarrow M/Q = \mathrm{Fi}_{22}$. In addition, in some cases where we obtained the natural homomorphism η, the computations of the normalizers of subgroup classes of D containing Q are too expensive, since the minimal permutation representation of Fi_{23} has degree 31 671. We make the following modification.

Choose a subgroup X of M. Using MAGMA, we explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$ and the degree of the action of W on the cosets is much smaller than that of M. We can now directly classify the radical p-subgroup classes of W, and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step (2). Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is p-local, then we apply Step (1) to K. If K is not p-local, we can replace M by K and repeat Step (2).

Steps (1) and (2) constitute the *modified local strategy*. After applying the strategy, possible fusions among the resulting list of radical subgroups can be decided readily by testing whether the subgroups in the list are pairwise Fi_{23}-conjugate.

As an example of the application of the modified strategy, consider the case where $p = 2$ and $M = 2^{11}.M_{23}$. We can find a subgroup X of index 23 in M, so that $W \simeq M_{23}$ and $K = O_2(M) = 2^{11}$. Applying the local strategy to W, we determine the radical subgroup classes of W, and the preimages are radical subgroups of M. In some cases, we must apply the modified local strategy to W. For example, when $p = 2$ and $M = 2.\mathrm{Fi}_{23}$, if we choose a subgroup X of M with index 3510, then $W \simeq \mathrm{Fi}_{22}$. We use the modified local strategy to classify the radical subgroup classes of W.

In our investigation, we used the minimal degree representation of Fi_{32} as a permutation group on 31 671 points. Its maximal subgroups were constructed using the details supplied in [5] and the black-box algorithms of Wilson [15]. We also made extensive use of the algorithm described in [6] to construct random elements, and the procedures described in [2] for deciding the conjectures.

The computations reported in this paper were carried out using MAGMA V.2.3-1 on a Sun UltraSPARC Enterprise 4000 server.

4 Radical subgroups and weights

Let $\Phi(G,p)$ be a set of representatives for conjugacy classes of radical subgroups of G. For $H, K \leq G$, we write $H \leq_G K$ if $x^{-1}Hx \leq K$; and write $H \in_G \Phi(G,p)$ if $x^{-1}Hx \in \Phi(G,p)$ for some $x \in G$. We shall follow the notation of [5]. In particular, if p is odd, then $p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p; if δ is $+$ or $-$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ. If X and Y are groups, we use $X.Y$ and $X:Y$ to denote an extension and a split extension of X by Y, respectively. Given a positive integer n, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the simple Fischer group Fi_{23}. Then

$$|G| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23,$$

and we may suppose $p \in \{2,3,5\}$, since both conjectures hold for a block with a cyclic defect group by Theorem 9.1 of [7].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|)$. Given $R \in \Phi(G,p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then (c.f. (1.3) of [2])

$$\mathcal{W}(B_0) = \sum_R |\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where R runs over the set $\Phi(G,p)$ such that the p-part $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\mathrm{Irr}^0(N/C(R)R)|$.

Lemma 4.1 *The non-trivial radical* 5*-subgroups R of* Fi_{23} *(up to conjugacy) are given in Table* 1*, where $F_{p^n}^m$ is the Frobenius group with kernel p^n and complement $\mathbb{Z}_m$.*

R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
5	$5 \times S_7$	$F_5^4 \times S_7$	
5^2	$5^2 \times 2$	$(F_5^4 \times F_5^4 \times 2).2.3$	16

Table 1: Non-trivial radical 5-subgroups of Fi_{23}

PROOF: By [5], p. 178, $G = \mathrm{Fi}_{23}$ has a unique class of elements x of order 5, and by MAGMA, $C(x) = 5 \times S_7$ and $N(\langle x \rangle) = F_5^4 \times S_7$. In addition, a Sylow 5-subgroup S of G is elementary abelian of order 25. We may suppose $x \in S$, so that $N_{N(\langle x \rangle)}(S) = F_5^4 \times F_5^4 \times 2$ and by MAGMA, $N(S) = (F_5^4 \times F_5^4 \times 2).2.3$. □

Lemma 4.2 *The non-trivial radical* 3*-subgroups* R *of* Fi_{23} *(up to conjugacy) are given in Table 2, where* $S \in \mathrm{Syl}_3(\mathrm{Fi}_{23})$ *is a Sylow* 3*-subgroup of* Fi_{23}*.*

R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
3	$3 \times O_7(3)$	$S_3 \times O_7(3)$	
3^6	3^6	$3^6\colon L_4(3)\colon 2$	2
3_+^{1+8}	3	$3_+^{1+8}:2_-^{1+6}\colon 3_+^{1+2}\colon 2S_4$	2
$3^3.3^6$	3^3	$3^3.3^6\colon (L_3(3)\times 2)$	2
$3_+^{1+8}.3$	3	$3_+^{1+8}.3.2_+^{1+4}.(S_3 \times S_3)$	4
$3^3.3.3^3.3^3$	3^3	$3^3.3.3^3.3^3\colon (L_3(3)\times 2)$	2
$3_+^{1+8}.3^2$	3	$3_+^{1+8}.3^2\colon (2S_4 \times 2)$	4
$3^3\colon 3.3^3.3^3.3^2$	3^2	$3^3\colon 3.3^3.3^3.3^2.(2S_4\times 2)$	4
$3_+^{1+8}\colon 3_+^{1+2}$	3	$3_+^{1+8}\colon 3_+^{1+2}.(2S_4\times 2)$	4
S	3	$S.2^3$	8

Table 2: Non-trivial radical 3-subgroups of Fi_{23}

Proof: Let $i \in \{1,\dots,5\}$, and let M_i denote a maximal 3-local subgroup of $G = \mathrm{Fi}_{23}$ where $M_1 = N(3A) \simeq S_3 \times O_7(3)$, $M_2 = N(3B) \simeq 3_+^{1+8}\colon 2_-^{1+6}\colon 3_+^{1+2}\colon 2S_4$, $M_3 = N(3^6) \simeq 3^6\colon L_4(3)\colon 2$, $M_4 = N(3B^3) \simeq 3^3.3^6\colon (L_3(3)\times 2)$ and $M_5 = N(3B^3) \simeq 3^3.[3^7]\colon (L_3(3)\times 2)$. By Theorem A of [14], each 3-local subgroup of G is G-conjugate to a subgroup of M_i for some i.

The subgroup M_1 and M_2 are normalizers of some $3A$ and $3B$ elements, so we can easily construct them in G. The centralizer $C(3C)$ of an element of class $3C$ is isomorphic to $3^6\colon (2 \times U_4(2))$ (c.f. [14], p. 78) and M_3 is the normalizer of $3^6 = O_3(C(3C))$ in G. As shown in the proof of Theorem A of [14] (see p. 81 of [14]), $O_3(M_2) = 3_+^{1+8}$ contains two classes of $3B$-pure elementary subgroups of order 3^3, with representatives say X and Y such that $M_4 = N(X)$ and $M_5 = N(Y)$. Repeated random selections of elements allow us to obtain X and Y; and so we can construct all of the subgroups M_i.

Let R be a non-trivial radical 3-subgroup of G. Then $N(R)$ is a 3-local subgroup of G, so that we may suppose $N(R) \le M_i$ for some i. Thus we may suppose $R \in \Phi(M_i, 3)$ such that $N(R) = N_{M_i}(R)$. We can apply the local strategy of [2] or the modified local strategy to each M_i.

(1) Let $3_+^{1+8} = O_3(M_2)$ and apply the local strategy to M_2. By Magma,

$$\Phi(M_2,3) = \{3_+^{1+8}, 3_+^{1+8}.3, 3_+^{1+8}.3^2, 3_+^{1+8}.3_+^{1+2}, S\}, \tag{4.2}$$

where $S \in \mathrm{Syl}_3(G)$. Moreover, $N(R) = N_{M_2}(R)$ for each $R \in \Phi(M_2,3)$ and we may suppose $\Phi(M_2,3) \subseteq \Phi(G,3)$. If $L = N_{M_2}(3_+^{1+8}.3)/3_+^{1+8}.3$, then by Magma, $|L| = 2^7\cdot 3^2$, $E = O_2(L)$ is an extra-special subgroup of order 2^5 and $C_L(E) = Z(E) = 2$. Since $L/E \le \mathrm{Out}(E)$, the outer automorphism group $\mathrm{Out}(E)$ contains a subgroup of order 9, so that E has plus type and $E = 2_+^{1+4}$. By Proposition 2.5.9 of [10], $\mathrm{Out}(2_+^{1+4}) = \mathrm{O}_4^+(2)$ has three subgroups of index 2, $S_3 \times S_3$, $S_3 \times S_3$ and $(S_3 \times 3).2$, and $(S_3\times 3).2/(3\times 3) \simeq 4$.

Using MAGMA, we have $L = 2_+^{1+4}.(S_3 \times S_3)$. If $R = S$, then $N(S)/S = 2^3$. The other normalizers of $R \in \Phi(M_2, 3)$ are given by the proofs (2) and (3) below.

(2) If $3^3.[3^7] = O_3(M_5)$, then by MAGMA, $[3^7] = 3.3^3.3^3$. Applying the local strategy of [2] to M_5, we have

$$\Phi(M_5, 3) = \{3^3.3.3^3.3^3, 3^3.3.3^3.3^3.3^2, 3_+^{1+8}{:}\, 3_+^{1+2}, S\}, \tag{4.3}$$

where $3_+^{1+8}{:}\, 3_+^{1+2} \in_G \Phi(M_2, 3)$. In addition, $N(R) = N_{M_5}(R)$ for all $R \in \Phi(M_5, 3)$, so we may suppose $\Phi(M_5, 3) \subseteq \Phi(G, 3)$. Since each subgroup $R/O_3(M_5)$ is a unipotent radical of a parabolic subgroup of $L_3(3)$, the structures of $N(R)$ can be determined using the subgroup structures of $L_3(3)$ (see p. 13 of [5]).

(3) Let $3^3.3^6 = O_2(M_4)$. Apply the local strategy of [2] to M_4. Then

$$\Phi(M_4, 3) = \{3^3.3^6, 3^3.3^6.3^2, 3_+^{1+8}{:}\, 3^2, S'\}, \tag{4.4}$$

where $S' \in \mathrm{Syl}_3(M_4)$ and $3_+^{1+8}{:}\, 3^2 \in_G \Phi(M_2, 3)$. In addition, $N(R) = N_{M_4}(R)$ for $R \in \Phi(M_4, 3) \backslash \{3^3.3^6.3^2, S'\}$. The structures of $N(R)$ can also be determined using the subgroup structures of $L_3(3)$. In particular, for $R \in \{3^3.3^6.3^2, S'\}$, by [5], p. 13,

$$N_{M_4}(R) = \begin{cases} 3^3.3^6.3^2.(2S_4 \times 2) & \text{if } R = 3^3.3^6.3^2, \\ S'.2^3 & \text{if } R = S'. \end{cases}$$

(4) Let $M = M_3$, $U = O_3(M)$ and $U \neq R \in \Phi(M, 3)$. Then $U \leq R$ and $D = R/U$ is a radical subgroup of $M/U = L_4(3){:}\, 2$, since $N_M(R)/U \simeq N_{M/U}(R/D)$. By [11], Lemma 2.1, $D = D \cap L_4(3)$ is a radical subgroup of $L_4(3)$. By the Borel-Tits theorem [4], $N_{L_4(3)}(D)$ is a parabolic subgroup Q of $L_4(3)$ and $D = O_3(Q)$. Thus Q is a subgroup of a maximal parabolic subgroup of $L_4(3)$. By [5], p. 69, we may suppose $Q \leq Q_1 = 3^3{:}\, L_3(3)$ (two classes of Q_1) or $Q \leq Q_2 = 3^4{:}\, 2(A_4 \times A_4).2$. If H is the commutator subgroup of M, then $H = 3^6{:}\, L_4(3)$. By MAGMA, $N_{M/U}(Q_1) = Q_1$ and $N_{M/U}(Q_2) = (Q_2).2$, so that an element of $L_4(3){:}\, 2 \backslash L_4(3)$ fuses the two classes of parabolic subgroups $3^3{:}\, L_3(3)$.

Let K_i be the preimage of $N_{L_4(3):2}(Q_i)$ in M. Then $K_1 = 3^6.3^3{:}\, L_3(3)$ and $K_2 = 3^6.3^4{:}\, 2(A_4 \times A_4).2.2$. We can apply the local strategy to each K_i to classify the radical subgroups of K_i. If $Q \leq 3^3{:}\, L_3(3)$, then $D \in_{L_4(3)} \Phi(3^3{:}\, L_3(3), 3)$ and $R \in_G \Phi(K_1, 3)$. If $Q \leq 3^4{:}\, 2(A_4 \times A_4).2$, then

$$N_{L_4(3):2}(D) \leq 3^4{:}\, 2(A_4 \times A_4).2.2 \leq L_4(3){:}\, 2.$$

Thus $D \in_{M/U} \Phi(3^4{:}\, 2(A_4 \times A_4).2.2, 3)$ and $R \in_G \Phi(K_2, 3)$ such that $N(R) = N_{K_2}(R)$. By MAGMA, $3^6.3^3 =_G 3^3.3^6 \in \Phi(M_4, 3)$ and

$$\Phi(3^6.3^3{:}\, L_3(3), 3) = \{3^3.3^6, 3^3.3^6.3^2, 3_+^{1+8}.3^2, S'\} =_G \Phi(M_4, 3). \tag{4.5}$$

Moreover, $N(R) \neq N_{M_3}(R)$ for all $R \in \Phi(K_1, 3)$, $N_{M_3}(R) = N_{K_1}(R)$ for $R = 3^3.3^6$ and $3_+^{1+8}.3^2$ and $N_{M_3}(R) \neq N_{K_1}(R)$ for $R = 3^6{:}\, 3^3{:}\, 3^2$ and S'. In addition,

$$N_{K_1}(R) = \begin{cases} 3^3.3^6{:}\, L_3(3) & \text{if } R = 3^3.3^6, \\ 3^3.3^6.3^2.2S_4 & \text{if } R = 3^3.3^6.3^2, \\ 3_+^{1+8}.3^2.2S_4 & \text{if } R = 3_+^{1+8}.3^2, \\ S'.2^2 & \text{if } R = S', \end{cases}$$

$N_{M_3}(3^3.3^6.3^2) = 3^3.3^6.3^2.(2S_4 \times 2)$ and $N_{M_3}(S') = S'.2^3$.

By MAGMA, $3^6.3^4 = O_3(K_2)$ is G-conjugate to $3_+^{1+8}.3$ and

$$\Phi(3^6.3^4{:}\,2(A_4 \times A_4).2.2, 3) = \{3_+^{1+8}.3, 3_+^{1+8}.3^2, S'\}, \tag{4.6}$$

where $S' \in_G \Phi(M_4, 3)$. Moreover, $N(3_+^{1+8}.3) = N_{M_3}(3_+^{1+8}.3) = N_{K_2}(3_+^{1+8}.3)$,

$$N(3_+^{1+8}.3^2) \neq N_{M_3}(3_+^{1+8}.3^2) = N_{K_2}(3_+^{1+8}.3^2) \simeq 3_+^{1+8}.3^2{:}\,2S_4$$

and $N(S') \neq N_{M_3}(S') = N_{K_2}(S') = S'.2^3$. It follows that

$$\Phi(M_3, 3) = \{3^6, 3^3.3^6, 3^3.3^6.3^2, 3_+^{1+8}.3, 3_+^{1+8}.3^2, S'\}, \tag{4.7}$$

and moreover, $N(R) = N_{M_3}(R)$ for $R \in \{3^6, 3_+^{1+8}.3\}$.

(5) Let $U = O_3(M_1) = 3$ and $U \neq R \in \Phi(M_1, 3)$. Then $C(U) = 3 \times O_7(3)$ is a index 2 subgroup of M_1 and we can apply the modified local strategy to $C(U)$. By [11], Lemma 2.1, $R = R \cap C(U)$ is a radical subgroup of $C(U)$, so R/U is a radical 3-subgroup of $C(U)/U = O_7(3)$. By the Borel-Tits theorem [4], $N_{O_7(3)}(R/U)$ is a parabolic subgroup of $O_7(3)$ which is contained in one of the maximal parabolic subgroups (c.f. [5], p. 109), $Q_1 = 3_+^{1+6}{:}\,(2A_4 \times A_4).2$, $Q_2 = 3^{3+3}{:}\,L_3(3)$ and $Q_3 = 3^5{:}\,U_4(2){:}\,2$. Let K_i be the preimage of Q_i in M_1. Then $K_1 = S_3 \times 3_+^{1+6}{:}\,(2A_4 \times A_4).2$, $K_2 = S_3 \times 3^{3+3}{:}\,L_3(3)$ and $K_3 = S_3 \times 3^5{:}\,U_4(2){:}\,2$. Moreover, we may suppose $R \in \Phi(K_i, 3)$ for some i such that $N(R) = N_{K_i}(R)$. The subgroups K_i can be constructed using the local structure of $O_7(3) = [\,C(U), C(U)\,]$ and we can apply the local strategy to each K_i.

Applying the local strategy to K_1, we have

$$\Phi(K_1, 3) = \{3 \times 3_+^{1+6}, 3 \times 3_+^{1+6}.3, 3 \times 3^5{:}\,3^3, S''\}, \tag{4.8}$$

where $3 \times 3^5{:}\,3^2$ is a radical subgroup of K_3 and $S'' \in \mathrm{Syl}_3(K_1)$. Moreover, $N(R) \neq N_{M_3}(R) = N_{K_1}(R)$ for all $R \in \Phi(K_1, 3)$ and (c.f. [5], p. 26)

$$N_{K_1}(R) = \begin{cases} S_3 \times 3_+^{1+6}(2A_4 \times A_4).2 & \text{if } R = 3 \times 3_+^{1+6}, \\ S_3 \times 3_+^{1+6}.3{:}\,2S_4 & \text{if } R = 3 \times 3_+^{1+6}.3, \\ S_3 \times 3^5{:}\,3^3{:}\,(S_4 \times 2) & \text{if } R = 3 \times 3^5{:}\,3^3, \\ S''.2^3 & \text{if } R = S''. \end{cases}$$

Applying the local strategy to K_2, we have

$$\Phi(K_2, 3) = \{3 \times 3^{3+3}, 3 \times 3^{3+3}{:}\,3^2, 3 \times 3_+^{1+6}.3, S''\}, \tag{4.9}$$

and moreover, $N(R) \neq N_{M_1}(R) = N_{K_2}(R)$ for all $R \in \Phi(K_2, 3)$, $N_{K_2}(R) \simeq N_{K_1}(R)$ for $R \in \{3 \times 3_+^{1+6}.3, S''\}$ and

$$N_{K_2}(R) = \begin{cases} S_3 \times 3^{3+3}{:}\,L_3(3) & \text{if } R = 3 \times 3^{3+3}, \\ S_3 \times 3^{3+3}.3^2{:}\,2S_4 & \text{if } R = 3 \times 3^{3+3}.3^2. \end{cases}$$

Similarly, by MAGMA,

$$\Phi(K_3, 3) = \{3^6, 3 \times 3^5{:}\,3^3, 3 \times 3^{3+3}{:}\,3^2, S''\}, \tag{4.10}$$

and moreover, $N(R) \neq N_{M_1}(R) = N_{K_3}(R)$ for all $R \in \Phi(K_3, 3)$, $N_{K_3}(R) \simeq N_{K_1}(R)$ for $R \in \{3 \times 3^5{:}\, 3^3, S''\}$ and $N_{K_3}(3 \times 3^{3+3}.3^2) \simeq N_{K_2}(3 \times 3^{3+3}.3^2)$. It follows that

$$\Phi(M_1, 3) = \{3, 3^6, 3 \times 3^{3+3}, 3 \times 3_+^{1+6}, 3 \times 3_+^{1+6}.3, 3 \times 3^{3+3}{:}\, 3^2, 3 \times 3^5{:}\, 3^3, S''\}, \quad (4.11)$$

and $N(R) \neq N_{M_1}(R)$ for all $R \in \Phi(M_1, 3)\backslash\{3\}$.

Thus the radical 3-subgroups are as claimed. The centralizers and the normalizers of R can be obtained by MAGMA. □

Lemma 4.3 *Given integer $1 \leq i \leq 6$, let M_i be the maximal subgroups of $G = \mathrm{Fi}_{23}$ such that $M_1 \simeq 2.\mathrm{Fi}_{22}$, $M_2 \simeq 2^2.U_6(2).2$, $M_3 \simeq 2^{11}.M_{23}$, $M_4 \simeq (2^2 \times 2_+^{1+8}).(3 \times U_4(2)).2$, $M_5 \simeq 2^{6+8}{:}\,(A_7 \times S_3)$ and $M_6 \simeq S_4 \times S_6(2)$. Suppose R is a non-trivial radical 2-subgroup of G. Then $N_G(R) \leq_G M_i$ for some i. In particular, if $N_G(R) \leq M_i$, then $N_G(R) = N_{M_i}(R)$ and $R \in_G \Phi(M_i, 2)$.*

PROOF: It follows by Flaass [9], Theorem (b), and [5], p. 177, that each M_i is a maximal subgroup of G. As shown in the proof of Section 6 of [9], each non-trivial radical subgroup R is contained in some M_i and $N_G(R) \leq M_i$. This completes the proof. □

Lemma 4.4 *The non-trivial radical 2-subgroups R of Fi_{23} (up to conjugacy) are given in Table 3, where $S \in \mathrm{Syl}_2(\mathrm{Fi}_{23})$ and H^* denotes a subgroup of G such that $H^* \simeq H$ and $H^* \neq_G H$.*

PROOF: If R is a non-trivial radical 2-subgroup of $G = \mathrm{Fi}_{23}$, then by Lemma 4.3, we may suppose $R \in \Phi(M_i, 2)$ such that $N(R) = N_{M_i}(R)$ for some $i = 1, \ldots, 6$.

(1) Let $M = M_6 \simeq S_4 \times S_6(2)$. Now S_4 has two classes of radical subgroups, $(2^2)^* = O_2(S_4)$ and $D_8 \in \mathrm{Syl}_2(S_4)$, and we may take $\Phi(S_4, 2) = \{(2^2)^*, D_8\}$. Note that $S_6(2)$ is the second derived group of M_6. We apply the modified local strategy to $S_6(2)$. Take a maximal subgroup K of $S_6(2)$ with index 28. MAGMA constructs the action of $S_6(2)$ on the cosets of K in $S_6(2)$ and the group homomorphism from $S_6(2)$ to the image H of the action. Then $H \simeq S_6(2)$ and the permutation representation of H has degree 28. Applying the local strategy of [2] to H, we learn that $S_6(2)$ has 7 classes of non-trivial radical 2-subgroups and their preimages are radical 2-subgroups (up to conjugacy) of $S_6(2)$. We may take

$$\Phi(S_6(2), 2) = \{1, 2^5, 2^6, 2^3.2^4, 2^5.2^3, 2^6.2^2, 2^3.2^4.2, S'\},$$

where $S' \in \mathrm{Syl}_2(S_6(2))$. It follows that

$$\Phi(S_4 \times S_6(2), 2) = \{R_1 \times R_2 \mid R_1 \in \Phi(S_4, 2), R_2 \in \Phi(S_6(2), 2)\}, \quad (4.12)$$

and moreover, by MAGMA, $N(R) = N_{M_6}(R)$ if and only if $R = (2^2)^*$ or $R = D_8$.

(2) Apply the local strategy to $M_5 = 2^{6+8}{:}\,(S_3 \times A_7)$. Then we may take

$$\Phi(M_5, 2) = \{2^{6+8}, 2^{6+8}.2, 2^{6+8}.2^2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.D_8, 2^{6+8}.2^3, 2^{11}.2^2.2^4, S\},$$

R	$C(R)$	$N/C(R)R$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
2	$2.\mathrm{Fi}_{22}$	1	
2^2	$2^2.U_6(2)$	2	
$(2^2)^*$	$2^2 \times S_6(2)$	S_3	
D_8	$2 \times S_6(2)$	1	
2^7	2^7	$S_6(2)$	1
2^{11}	2^{11}	M_{23}	2
$2^2 \times 2_+^{1+8}$	2^3	$(3 \times U_4(2)).2$	1
$(2^2 \times 2_+^{1+8}).2$	2^2	S_6	1
$2^{11}.2^3$	2^4	$L_3(2)$	1
2^{6+8}	2^6	$S_3 \times A_7$	0
$2^{6+8}.2$	2^6	A_7	0
$2^{11}.2^4$	2^3	$3.S_5$	1
$2^{6+8}.2^2$	2^3	$S_3 \times S_3$	1
$(2^2 \times 2_+^{1+8}).2.2^4$	2^3	$3.(S_3 \times S_3)$	4
$2^{11}.2^2.2^3$	2^2	S_3	1
$2^{11}.2^2.2^4$	2^3	$F_{3^2}^2$	4
$2^{6+8}.D_8$	2^2	S_3	1
$2^{6+8}.2^3$	2^3	S_3	1
S	2^2	1	1

Table 3: Non-trivial radical 2-subgroups of Fi_{23}

and $N_{M_5}(R) = N(R)$ for each $R \in \Phi(M_5, 2)$. We may suppose $\Phi(M_5, 2) \subseteq \Phi(G, 2)$.

(3) Apply the local strategy to $M_4 = (2^2 \times 2_+^{1+8})(3 \times U_4(2)).2$. Then we may take

$$\Phi^*(M_4, 2) = \{(2^2 \times 2_+^{1+8}).2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^4, 2^{11}.2^2.2^3, 2^{6+8}.D_8, 2^{11}.2^2.2^4\},$$

where $\Phi^*(M_4, 2) = \Phi(M_4, 2) \backslash \{2^2 \times 2_+^{1+8}, S\}$ and in addition, $N_{M_4}(R) = N(R)$ for each $R \in \Phi(M_4, 2)$. We may suppose $\Phi(M_4, 2) \subseteq \Phi(G, 2)$.

(4) Apply the modified local strategy to $M = M_3 = 2^{11}.M_{23}$. Take K a maximal subgroup of M with index 23. Using MAGMA, we compute the action of M on the cosets of K in M and a homomorphism η from M to the image H of the action. Then H is isomorphic to M_{23} and is described by the permutation representation of degree 23. By applying the local strategy to H, we classify the radical subgroups of H. The preimages of radical subgroups of H are radical subgroups R of M and, moreover, $N_M(R)$ is the preimage of $N_H(\eta(R))$. Thus we may take

$$\Phi(M_3, 2) = \{2^{11}, 2^{11}.2^3, 2^{11}.2^4, 2^{6+8}.2, 2^{11}.2^2.2^3, 2^{6+8}.2^3, 2^{11}.2^2.2^4, S\},$$

and moreover, $N(R) = N_M(R)$ for each $R \in \Phi(M, 2)$. We may suppose $\Phi(M_3, 2) \subseteq \Phi(G, 2)$.

(5) Let $M = M_2 = 2^2.U_6(2).2$. We apply the modified local strategy to the maximal subgroup M.

Using MAGMA, we find a subgroup K of M with index 693, and then get the image H of the action of M on the cosets of K in M and the group homomorphism η from M to H. Now H of degree 693 is isomorphic to $U_6(2).2$, and the commutator group $H' \simeq U_6(2)$. Then the preimages of radical 2-subgroups of H are the radical 2-subgroups of M. So we need only to classify the radical 2-subgroups of H.

If D is a non-trivial radical subgroup, then $D \cap H'$ is a radical subgroup of H'. If $D \cap H' = 1$, then $D = 2$, $N_H(D) = 2 \times S_6(2)$ and $\eta^{-1}(D) =_G D_8 \in \Phi(M_6, 2)$. Suppose $D \cap H'$ is non-trivial. It follows by the Borel-Tits theorem that D is H-conjugate to a radical subgroup of $K_1 \simeq 2_+^{1+8}.U_4(2).2$, $K_2 = 2^{4+8}{:}\,(3 \times A_5){:}\,2.2$ or $K_3 = 2^9{:}\,L_3(4){:}\,2$ (see [5], p. 115), and in addition, $N_H(D) = N_{K_i}(D)$ for $D \leq K_i$.

If $MK_i = \eta^{-1}(K_i)$ for $i = 1, 2, 3$, then $MK_1 \simeq (2^2 \times 2_+^{1+8}).U_4(2).2 \leq_G M_4$, $MK_2 \simeq 2^{6+8}(3 \times A_5).2.2 \leq_G M_5$ and $MK_3 \simeq 2^{11}.L_3(4).2 \leq_G M_3$. The preimages of the radical subgroups of K_i are the radical subgroups of MK_i. Applying the local strategy to each K_i, we can classify its radical subgroups. If $\Phi^*(MK_i, 2) = \Phi(MK_i, 2) \backslash \{O_2(MK_i), S\}$, then we may take

$$\begin{aligned}
\Phi^*(MK_1, 2) &= \{(2^2 \times 2_+^{1+8}).2, 2^{11}.2^4, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^3, 2^{11}.2^2.2^4, 2^{6+8}.D_8\},\\
\Phi^*(MK_2, 2) &= \{2^{6+8}.2, 2^2.2^{4+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.2^2, 2^{11}.2^2.2^4, 2^{6+8}.D_8\},\\
\Phi^*(MK_3, 2) &= \{2^{11}.2, 2^{11}.2^4, 2^{6+8}.2, 2^{6+8}.2^2, 2^{11}.2^2.2^3, 2^{11}.2^2.2^4\}.
\end{aligned}$$

It follows that we may take

$$\begin{aligned}
\Phi(M_2, 2) = \{&2^2, D_8, 2^{11}, 2^2 \times 2_+^{1+8}, (2^2 \times 2_+^{1+8}).2, 2^2.2^{4+8}.2, 2^{11}.2, 2^{6+8},\\
&2^{11}.2^4, 2^{6+8}.2, 2^{6+8}.2^2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^3, 2^{11}.2^2.2^4, 2^{6+8}.D_8, S\},
\end{aligned}$$

and moreover, for each $R \in \Omega = \{2^2, D_8, (2^2 \times 2_+^{1+8}).2, 2^{11}.2^2.2^3, 2^{6+8}.D_8, S\}$ we have $N_{M_2}(R) = N(R)$. In addition, for $R \in \Phi(M_2, 2) \backslash \Omega$,

$$N_{M_2}(R)/R = \begin{cases}
L_3(4).2 & \text{if } R = 2^{11},\\
U_4(2).2 & \text{if } R = 2^2 \times 2_+^{1+8},\\
S_3 \times S_3 & \text{if } R = 2^2.2^{4+8}.2,\\
L_3(2) & \text{if } R = 2^{11}.2,\\
(3 \times A_5).2.2 & \text{if } R = 2^{6+8},\\
S_5 & \text{if } R = 2^{11}.2^4,\\
S_5 & \text{if } R = 2^{6+8}.2,\\
S_3 & \text{if } R = 2^{6+8}.2^2,\\
S_3 \times S_3 & \text{if } R = (2^2 \times 2_+^{1+8}).2.2^4,\\
S_3 & \text{if } R = 2^{11}.2.2^4.
\end{cases}$$

(6) Let $M = M_1 = 2.\mathrm{Fi}_{22}$. The radical subgroups of M can be classified using the modified local strategy. We first find a subgroup K of M with index 3510, and then get the image H of the action of M on the cosets of K in M and the group

homomorphism η from M to H. Then H of degree 3510 is isomorphic to Fi_{22}. As shown in the proof of the main theorem of [9], a non-trivial radical 2-subgroup D of H is conjugate to a radical 2-subgroup of K_i and $N_H(D) \leq_H K_i$, where $1 \leq i \leq 5$ and K_i are maximal subgroups of H such that $K_1 \simeq 2.U_6(2)$, $K_2 \simeq 2^{10}{:}\, M_{22}$, $K_3 \simeq 2^6{:}\, S_6(2)$, $K_4 \simeq (2 \times 2_+^{1+8}{:}\, U_4(2)){:}\, 2$ and $K_5 \simeq 2^{5+8}{:}\, (S_3 \times A_6)$. Applying the modified local strategy to each maximal subgroup K_i of H, we get all the radical 2-subgroups of K_i and hence those of $H = \mathrm{Fi}_{22}$; these are listed in Table 4.

R	$C_H(R)$	$N_H(R)$	$\lvert\mathrm{Irr}^0(N_H(R)/C_H(R)R)\rvert$
2	$2.U_6(2)$	$2.U_6(2)$	
2^6	2^6	$2^6{:}\, S_6(2)$	1
2^{10}	2^{10}	$2^{10}{:}\, M_{22}$	0
$2 \times 2_+^{1+8}$	2^2	$(2 \times 2_+^{1+8}{:}\, U_4(2)){:}\, 2$	0
$(2 \times 2_+^{1+8}).2$	2	$(2 \times 2_+^{1+8}).2.S_6$	1
$2^{10}.2^3$	2^3	$2^{10}.2^3.L_3(2)$	1
2^{5+8}	2^5	$2^{5+8}.(S_3 \times A_6)$	2
$2^{5+8}.2$	2^5	$2^{5+8}.2.A_6$	2
$2^{10}.2^4$	2^2	$2^{10}.2^4.S_5$	1
$2^{5+8}.2^2$	2^2	$2^{5+8}.2^2.(S_3 \times S_3)$	1
$(2 \times 2_+^{1+8}).2.2^4$	2^2	$(2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3)$	1
$2^{10}.2^2.2^3$	2	$2^{10}.2^2.2^3.S_3$	1
$2^{10}.2^2.2^4$	2^2	$2^{10}.2^2.2^4.S_3$	1
$2^{5+8}.D_8$	2	$2^{5+8}.D_8.S_3$	1
$2^{5+8}.2^3$	2^2	$2^{5+8}.2^3.S_3$	1
$2^{5+8}.D_8.2$	2	$2^{5+8}.D_8.2$	1

Table 4: Non-trivial radical 2-subgroups of Fi_{22}

Since the preimages of radical subgroups of H are radical subgroups of M_1, we may take

$$\begin{aligned}\Phi^*(M_1, 2) \quad = \quad & \{2^2, 2^7, 2^{11}, 2^2 \times 2_+^{1+8}, (2^2 \times 2_+^{1+8}).2, 2^{11}.2^3, 2^{6+8}, 2^{6+8}.2, \\ & 2^{11}.2^4, 2^{6+8}.2^2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^3, 2^{11}.2^2.2^4, 2^{6+8}.D_8, 2^{6+8}.2^3\},\end{aligned}$$

and moreover, for $R \in \{2, 2^7, (2^2 \times 2_+^{1+8}).2, 2^{11}.2^3, 2^{6+8}.2^2, 2^{11}.2^2.2^3, 2^{6+8}.D_8, 2^{6+8}.2^3, S\}$, $N_{M_1}(R) = N_G(R)$.

This completes the classification of radical 2-subgroups of G. The centralizers and normalizers of $R \in \Phi(G, 2)$ are given by MAGMA. $\square$

Lemma 4.5 *Let $G = \mathrm{Fi}_{23}$, and let $\mathrm{Blk}^0(G, p)$ be the set of p-blocks with a non-trivial defect group and $\mathrm{Irr}^+(G)$ the characters of $\mathrm{Irr}(G)$ with positive p-defect.*

(a) *If* $p=5$, *then* $\mathrm{Irr}^0(G,p)=\{B_i \mid 0\le i\le 6\}$ *such that* $D(B_0)=_G D(B_1)\simeq 5^2$ *and* $D(B_i)\simeq 5$ *for* $2\le i\le 6$, *where* $B_0=B_0(G)$ *is the principal block of* G *and* $D(B)$ *is a defect group of a block* B. *In the notation of* [5], *p.* 178,

$$\mathrm{Irr}(B_i)=\begin{cases}\{\chi_{11},\chi_{19},\chi_{25},\chi_{34},\chi_{39}\} & \text{if } i=2,\\ \{\chi_{13},\chi_{24},\chi_{40},\chi_{43},\chi_{49}\} & \text{if } i=3,\\ \{\chi_{21},\chi_{55},\chi_{60},\chi_{79},\chi_{86}\} & \text{if } i=4,\\ \{\chi_{22},\chi_{68},\chi_{87},\chi_{91},\chi_{93}\} & \text{if } i=5,\\ \{\chi_{23},\chi_{33},\chi_{59},\chi_{70},\chi_{78}\} & \text{if } i=6,\end{cases}$$

and in addition, $\mathrm{Irr}(B_1)=\{\chi_2,\chi_6,\chi_7,\chi_8,\chi_9,\chi_{14},\chi_{27},\chi_{32},\chi_{37},\chi_{44},\chi_{48},\chi_{57},\chi_{64},\chi_{69},\chi_{73},\chi_{76},\chi_{82},\chi_{89},\chi_{92},\chi_{96}\}$ *and* $\mathrm{Irr}(B_0)=\mathrm{Irr}^+(G)\backslash(\cup_{i=1}^6\mathrm{Irr}(B_i))$. *Moreover,* $\ell(B_i)=4$ *for* $2\le i\le 6$ *and* $\ell(B_j)=16$ *for* $j=0,1$.

(b) *If* $p=3$, *then* $\mathrm{Blk}(G,2)=\{B_0,B_1\}$ *such that* $D(B_1)\simeq 3$. *In the notation of* [5], *p.* 178, $\mathrm{Irr}(B_1)=\{\chi_{69},\chi_{82},\chi_{93}\}$ *and* $\mathrm{Irr}(B_0)=\mathrm{Irr}^+(G)\backslash\mathrm{Irr}(B_1)$. *Moreover,* $\ell(B_1)=2$ *and* $\ell(B_0)=32$.

(c) *If* $p=2$, *then* $\mathrm{Blk}(G,2)=\{B_0,B_1,B_2\}$ *such that* $D(B_1)\simeq 2$ *and* $D(B_2)\simeq D_8$. *In the notation of* [5], *p.* 178, $\mathrm{Irr}(B_1)=\{\chi_{56},\chi_{57}\}$,

$$\mathrm{Irr}(B_2)=\{\chi_{60},\chi_{64},\chi_{65},\chi_{76},\chi_{77}\}$$

and $\mathrm{Irr}(B_0)=\mathrm{Irr}^+(G)\backslash(\mathrm{Irr}(B_1)\cup\mathrm{Irr}(B_2))$. *Moreover,* $\ell(B_1)=1$, $\ell(B_2)=2$ *and* $\ell(B_0)=20$.

PROOF: If $B\in\mathrm{Blk}(G,p)$ is non-principal with $D=D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By Lemmas 4.1, 4.2 and 4.4, $D\in_G\{5,5^2,3,2,D_8\}$, and moreover, if $Q\in\{5^2,3,2,D_8\}$, then G has exactly one non-principal block B with $D(B)=_G Q$, since $|\mathrm{Irr}^0(C(Q)Q/Q)|=1$. If $D=5$, then $|\mathrm{Irr}^0(C(D)D/D)|=5$, so G has exactly 5 blocks, B_i for $2\le i\le 6$ with $D(B_i)=_G D$.

Using the method of central characters, $\mathrm{Irr}(B)$ is as above. If $D(B)$ is cyclic or isomorphic to D_8, then $\ell(B)$ is the number of B-weights (see [7] and [13]), so that

$$\ell(B_i)=\begin{cases}4 & \text{if } p=5 \text{ and } i\ge 2,\\ 2 & \text{if } p=3 \text{ and } i=1,\\ 1 & \text{if } p=2 \text{ and } i=1,\\ 2 & \text{if } p=2 \text{ and } i=2.\end{cases}$$

Suppose $p=5$ and $B=B_1$. Since all non-trivial elements of $D(B)$ are G-conjugate, it follows by a theorem of Brauer that $\mathrm{k}(B)=\ell(B)+\ell(b)$, where b is a block of $C_G(5)=5\times S_7$ such that $b^G=B$. Thus $b=b_0\times b_1$, where $b_0=B_0(5)$ and $b_1\in\mathrm{Blk}(S_7)$. So $D(b_1)\simeq 5$ and $\mathcal{W}(b_1)=\ell(b_1)$. But $D(b_1)\in\mathrm{Syl}_5(S_7)$, so $N_{S_7}(D(b_1))\simeq 5{:}4\times 2$. It follows that $\mathcal{W}(b_1)=4$ and $\ell(b)=\ell(b_1)=4$, so that $\ell(B)=20-4=16$.

If $\ell(G)$ is the number of p-regular G-conjugacy classes, then $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell(G)=\bigcup_{B\in\mathrm{Blk}^0(G,p)}\ell(B)+|\mathrm{Irr}^0(G)|.$$

This completes the proof. □

Theorem 4.6 *Let $G = \mathrm{Fi}_{23}$ and let B be a p-block of G with a non-cyclic defect group. Then the number of B-weights is the number of irreducible Brauer characters of B.*

PROOF: If $B = B_0$, then the proof of Theorem 4.6 follows by Lemmas 4.1, 4.2, 4.4, 4.5 and (4.1). Suppose $B \neq B_0$. Then $p = 5$ and $D(B) =_G 5^2$, so we may suppose $D(B) = 5^2$. By MAGMA, $N(5^2)/5^2$ has 32 irreducible characters and by Lemma 4.1, 16 of them are the B_0-weight characters, so that $\mathcal{W}(B) = 32 - 16 = 16 = \ell(B)$. □

5 Radical Chains of Fi_{23}

Let $G = \mathrm{Fi}_{23}$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$.

Lemma 5.1 *In the notation of Lemma* 4.1*, the radical* 5*-chains C of G (up to conjugacy) are given in Table* 5.

C		$N(C)$	C		$N(C)$
$C(1)$	1	Fi_{23}	$C(2)$	$1 < 5$	$F_5^4 \times S_7$
$C(3)$	$1 < 5 < 5^2$	$F_5^4 \times F_5^4 \times 2$	$C(4)$	$1 < 5^2$	$(F_5^4 \times F_5^4 \times 2).2.3$

Table 5: Radical 5-chains of Fi_{23}

PROOF: Straightforward. □

Lemma 5.2 (a) *In the notation of Lemma* 4.2 *and* (4.2)-(4.11)*, the radical* 3*-chains $C(i)$ for $1 \leq i \leq 24$ and their normalizers are given in Table* 6.

(b) *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that $\mathcal{R}^0(G)/G = \{C(i) : 1 \leq i \leq 24\}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d) \qquad (5.1)$$

for all integers $d \geq 0$.

PROOF: (b) Suppose C' is a radical chain such that

$$C' : 1 < P'_1 < \ldots < P'_m. \qquad (5.2)$$

Let $C \in \mathcal{R}(G)$ be given by (2.1) with $P_1 \in \Phi(G, 3)$.

C		$N(C)$
$C(1)$	1	Fi_{23}
$C(2)$	$1 < 3$	$S_3 \times O_7(3)$
$C(3)$	$1 < 3 < 3^6$	$S_3 \times 3^5{:}\, U_4(2){:}\, 2$
$C(4)$	$1 < 3 < 3 \times 3^{3+3} < 3 \times 3^{3+3}{:}\, 3^2$	$S_3 \times 3^{3+3}{:}\, 3^2{:}\, 2S_4$
$C(5)$	$1 < 3 < 3 \times 3^{3+3}$	$S_3 \times 3^{3+3}{:}\, L_3(3)$
$C(6)$	$1 < 3 < 3 \times 3_+^{1+6} < 3 \times 3_+^{1+6}.3$	$S_3 \times 3_+^{1+6}.3 : 2S_4$
$C(7)$	$1 < 3 < 3 \times 3_+^{1+6}$	$S_3 \times 3_+^{1+6}(2A_4 \times A_4).2$
$C(8)$	$1 < 3 < 3 \times 3_+^{1+6} < 3 \times 3^5{:}\, 3^3$	$S_3 \times 3^5{:}\, 3^3{:}\, (S_4 \times 2)$
$C(9)$	$1 < 3 < 3 \times 3_+^{1+6} < 3 \times 3_+^{1+6}.3 < S''$	$S''.2^3$
$C(10)$	$1 < 3_+^{1+8}$	$3_+^{1+8}{:}\, 2_-^{1+6}{:}\, 3_+^{1+2}{:}\, 2S_4$
$C(11)$	$1 < 3^3.3^6 < 3^3.3^6.3^2$	$3^3.3^6{:}\, 3^2{:}\, (2S_4 \times 2)$
$C(12)$	$1 < 3^3.3^6$	$3^3.3^6{:}\, (L_3(3) \times 2)$
$C(13)$	$1 < 3^3.3^6 < 3_+^{1+8}.3^2$	$3_+^{1+8}.3^2{:}\, (2S_4 \times 2)$
$C(14)$	$1 < 3^6$	$3^6{:}\, L_4(3){:}\, 2$
$C(15)$	$1 < 3^6 < 3^3.3^6$	$3^3.3^6{:}\, L_3(3)$
$C(16)$	$1 < 3^6 < 3^3.3^6 < 3^3.3^6{:}\, 3^2$	$3^3{:}\, 3^6{:}\, 3^2{:}\, 2S_4$
$C(17)$	$1 < 3^6 < 3^3.3^6.3^2$	$3^3{:}\, 3^6{:}\, 3^2{:}\, (2S_4 \times 2)$
$C(18)$	$1 < 3^6 < 3^3.3^6.3^2 < S'$	$S'.2^3$
$C(19)$	$1 < 3^6 < 3_+^{1+8}.3$	$3_+^{1+8}.3.2_+^{1+4}.(S_3 \times S_3)$
$C(20)$	$1 < 3^6 < 3_+^{1+8}.3 < 3_+^{1+8}.3^2$	$3_+^{1+8}{:}\, 3^2{:}\, 2S_4$
$C(21)$	$1 < 3^6 < 3_+^{1+8}.3 < 3_+^{1+8}.3^2 < S'$	$S'.2^2$
$C(22)$	$1 < 3^6 < 3_+^{1+8}.3 < S'$	$S'.2^3$
$C(23)$	$1 < 3^3.3.3^3.3^3 < 3_+^{1+8}.3_+^{1+2}$	$3_+^{1+8}.3_+^{1+2}{:}\, (2S_4 \times 2)$
$C(24)$	$1 < 3^3.3.3^3.3^3$	$3^3.3.3^3.3^3{:}\, (L_3(3) \times 2)$

Table 6: Radical 3-chains of Fi_{23}

Case (1). We first consider the radical subgroups of G contained in M_2. Let $R \in \Phi(M_2, 3)\backslash\{3_+^{1+8}\}$. Define G-invariant subfamilies $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{M}^+(R)/G &= \{C' \in \mathcal{R}/G : P_1' = R\}, \\ \mathcal{M}^0(R)/G &= \{C' \in \mathcal{R}/G : P_1' = 3_+^{1+8}, P_2' = R\}. \end{aligned} \tag{5.3}$$

For $C' \in \mathcal{M}^+(R)$ given by (5.2), the chain

$$g(C') : 1 < 3_+^{1+8} < P_1' = R < P_2' < \ldots < P_m' \tag{5.4}$$

is a chain in $\mathcal{M}^0(R)$ and $N(C') = N(g(C'))$. For $B \in \mathrm{Blk}(G)$ and for integer $d \geq 0$,

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(g(C')), B, d). \tag{5.5}$$

In addition, g is a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$. So we may suppose

$$C \notin \bigcup_{R\in\Phi(M_2,3)\setminus\{3_+^{1+8}\}} (\mathcal{M}^+(R)\cup\mathcal{M}^0(R)).$$

Thus $P_1 \notin \{3_+^{1+8}.3, 3_+^{1+8}.3^2, 3_+^{1+8}.3_+^{1+2}, S\}$, and if $P_1 = 3_+^{1+8}$, then $C =_G C(10)$. We may assume

$$P_1 \in \{3, 3^6, 3^3.3^6, 3^3.3.3^3.3^3, 3^3{:}\,3.3^3.3^3.3^2\} \subseteq \Phi(G,3).$$

Case (2). Let $\Omega = \{3\times 3^5{:}\,3^3, 3\times 3^{3+3}{:}\,3^2, S''\} \subseteq \Phi(M_1,3)$ and suppose $Q\in\Omega$. By (4.10), we may suppose $\Omega \subseteq \Phi(N_{M_1}(3^6),3)$, and moreover, $N_{M_1}(Q) = N_{N_{M_1}(3^6)}(Q)$. Define G-invariant subfamilies $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned}\mathcal{L}^+(Q)/G &= \{C'\in\mathcal{R}/G : P_1' = 3, P_2' = Q\},\\ \mathcal{L}^0(Q)/G &= \{C'\in\mathcal{R}/G : P_1' = 3, P_2' = 3^6, P_3' = Q\}.\end{aligned} \tag{5.6}$$

A similar proof to Case (1) shows that there exists a bijection g between $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ such that $N(C') = N(g(C'))$ for each $C'\in\mathcal{L}^+(Q)$. Thus we may suppose

$$C \notin \bigcup_{Q\in\Omega} (\mathcal{L}^+(Q)\cup\mathcal{L}^0(Q)). \tag{5.7}$$

It follows that if $P_1 = 3$, then we may assume $P_2 \in \Phi(M_1,3)\setminus\Omega$ and if, moreover, $P_2 = 3^6$, then $C =_G C(3)$.

By (4.11), we may suppose $P_2 \in \{3\times 3^{3+3}, 3\times 3_+^{1+6}, 3\times 3_+^{1+6}.3\}$. By (4.9), we may suppose $3\times 3_+^{1+6}.3 \in \Phi(N_{M_1}(3\times 3^{3+3}),3)$, and in addition, $N_{M_1}(3\times 3_+^{1+6}.3) \leq N_{M_1}(3\times 3^{3+3})$. Let $\mathcal{L}^+(3\times 3_+^{1+6}.3)$ and $\mathcal{L}^0(3\times 3_+^{1+6}.3)$ be defined by (5.6) with Q replaced by $3\times 3_+^{1+6}.3$ and 3^6 by $3\times 3^{3+3}$. A similar proof shows that we may suppose

$$C \notin (\mathcal{L}^+(3\times 3_+^{1+6}.3)\cup\mathcal{L}^0(3\times 3_+^{1+6}.3)),$$

so we may assume $P_2 \neq_G 3\times 3_+^{1+6}.3$ and if $P_2 = 3\times 3^{3+3}$, then $P_3 \neq_G 3\times 3_+^{1+6}.3$.

Let $C' : 1 < 3 < 3\times 3^{3+3} < S''$ and $g(C') : 1 < 3 < 3\times 3^{3+3} < 3\times 3^{3+3}{:}\,3^2 < S''$. Then $N(C') = N(g(C'))$ and we may delete C' and $g(C')$. Similarly, we can delete $C' : 1 < 3 < 3\times 3_+^{1+6} < S''$ and $g(C') : 1 < 3 < 3\times 3_+^{1+6} < 3\times 3^5{:}\,3^3 < S''$. It follows that if $P_1 = 3$ and $P_2 =_G 3\times 3^{3+3}$ or $3\times 3_+^{1+6}$, then $C \in_G \{C(4), C(5), C(6), C(7), C(8), C(9)\}$ and we may suppose

$$P_1 \in \{3^6, 3^3.3^6, 3^3.3.3^3.3^3, 3^3{:}\,3.3^3.3^3.3^2\} \subseteq \Phi(G,3).$$

Case (3). Let $C' : 1 < 3^3.3^6 < S'$ and $g(C') : 1 < 3^3.3^6 < 3^3.3^6{:}\,3^2 < S'$. By the proof (3) of Lemma 4.2, $N(C') = N(g(C'))$ and we may delete C' and $g(C')$. Thus if $P_1 = 3^3.3^6$, then $C \in_G \{C(11), C(12), C(13)\}$ or $C =_G C(1)' : 1 < 3^3.3^6 < 3_+^{1+8}{:}\,3^2 < S'$.

Case (4). Let $C' = C(1)'$ and $g(C') : 1 < 3^6 < S'$. Then $N(C') = N(g(C')) = S'.2^3$ and (5.5) holds. Let $\mathcal{L}^+(3_+^{1+8}.3^2)$ and $\mathcal{L}^0(3_+^{1+8}.3^2)$ be defined as (5.6) with Q replaced by $3_+^{1+8}.3^2$, 3 by 3^6 and 3^6 by $3^3.3^6$. Then (5.7) holds with $Q = 3_+^{1+8}.3^2$. If $C' : 1 < 3^6 < 3^3.3^6 < S'$ and $g(C') : 1 < 3^6 < 3^3.3^6 < 3^6.3^3.3^2 < S'$, then

$N(C') = N(g(C')) = S'.2^2$ and (5.5) holds. It follows by (4.5) and (4.6) that if $P_1 = 3^6$, then $C \in_G \{C(k) : 14 \leq k \leq 22\}$.

Case (5). Let $\mathcal{M}^+(3^3.3.3^3.3^3.3^2)$ and $\mathcal{M}^0(3^3.3.3^3.3^3.3^2)$ be defined by (5.3) with R replaced by $3^3.3.3^3.3^3.3^2$ and 3_+^{1+8} by $3^3.3.3^3.3^3$. Then for each $C' \in \mathcal{M}^+(3^3.3.3^3.3^3.3^2)$ (5.5) holds, so that we may suppose $P_1 \neq_G 3^3.3.3^3.3^3.3^2$ and moreover, if $P_1 = 3^3.3.3^3.3^3$, then $P_2 \neq_G 3^3.3.3^3.3^3.3^2$. Let $C' : 1 < 3^3.3.3^3.3^3 < S$ and $g(C') : 1 < 3^3.3.3^3.3^3 < 3_+^{1+8}.3_+^{1+2} < S$. Then $N(C') = N(g(C')) = N(S)$ and (5.5) holds. If $P_1 = 3^3.3.3^3.3^3$, then by (4.3), $C \in \{C(23), C(24)\}$.

The proof of (a) follows easily by that of (b) or Lemma 4.2. $\square$

Lemma 5.3 (a) *In the notation of Lemma* 4.4 *and its proof, the radical* 2*-chains* $C(i)$ *for* $1 \leq i \leq 32$ *and their normalizers are given in Table* 7.

(b) *Let* $\mathcal{R}^0(G)$ *be the* G*-invariant subfamily of* $\mathcal{R}(G)$ *such that* $\mathcal{R}^0(G)/G = \{C(i) : 1 \leq i \leq 32\}$. *Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d)$$

for all integers $d \geq 0$.

The proof of Lemma 5.3 is similar to that of Lemma 5.2 and is in Appendix A.

6 The proof of Dade's Conjecture

Let $N(C)$ be the normalizer of a radical p-chain C. If $N(C)$ is a maximal subgroup of Fi_{23}, then the character table of $N(C)$ can be found in the library of character tables distributed with GAP. If this is not the case, we construct a "useful" description of $N(C)$ and attempt to compute directly its character table using MAGMA.

If $N(C)$ is soluble, we construct a power-conjugate presentation for $N(C)$ and use this presentation to obtain the character table.

If $N(C)$ is insoluble, we construct faithful representations for $N(C)$ and use these as input to the character table construction function. We employ two strategies to obtain faithful representations of $N(C)$.

1. Construct the actions of $N(C)$ on the cosets of soluble subgroups of $N(C)$.

2. Construct the orbits of $N(C)$ on the underlying set of Fi_{23}; for the stabiliser of an orbit representative, construct the action of $N(C)$ on its cosets.

In several cases, however, none of the representations constructed was of sufficiently small degree to allow us to construct the required character table.

In these cases, we directly calculate the character table of $N(C)$ as follows: first calculate the character tables of some subgroups and quotient groups of $N(C)$; next induce or lift these characters to $N(C)$, so the liftings and the irreducible characters from the induction form a partial character table T of $N(C)$; finally decompose the remaining inductions or the tensor products of the inductions using the table T.

The tables listing degrees of irreducible characters referenced in the proof of Theorem 6.1 are in Appendix C.

C		$N(C)$
$C(1)$	1	Fi_{23}
$C(2)$	$1 < 2$	$2.\mathrm{Fi}_{22}$
$C(3)$	$1 < 2 < 2^{11}$	$2^{11}.M_{22}$
$C(4)$	$1 < 2^{11}$	$2^{11}.M_{23}$
$C(5)$	$1 < 2 < 2^2 \times 2_+^{1+8}$	$(2^2 \times 2_+^{1+8}).U_4(2).2$
$C(6)$	$1 < 2^2 \times 2_+^{1+8}$	$(2^2 \times 2_+^{1+8}).(3 \times U_4(2)).2$
$C(7)$	$1 < 2 < 2^{6+8}$	$2^{6+8}.(S_3 \times A_6)$
$C(8)$	$1 < 2^{6+8}$	$2^{6+8}.(S_3 \times A_7)$
$C(9)$	$1 < 2^{6+8} < 2^{6+8}.2$	$2^{6+8}.2.A_7$
$C(10)$	$1 < 2 < 2^{6+8} < 2^{6+8}.2$	$2^{6+8}.2.A_6$
$C(11)$	$1 < 2 < 2^{6+8} < (2^2 \times 2_+^{1+8})2.2^4 < 2^{11}.2^2.2^4$	$2^{11}.2^2.2^4.S_3$
$C(12)$	$1 < 2^{6+8} < (2^2 \times 2_+^{1+8})2.2^4 < 2^{11}.2^2.2^4$	$2^{11}.2^2.2^4.F_{3^2}^2$
$C(13)$	$1 < 2^{6+8} < (2^2 \times 2_+^{1+8})2.2^4$	$(2^2 \times 2_+^{1+8})2.2^4.3.(S_3 \times S_3)$
$C(14)$	$1 < 2 < 2^{6+8} < (2^2 \times 2_+^{1+8})2.2^4$	$(2^2 \times 2_+^{1+8})2.2^4.(S_3 \times S_3)$
$C(15)$	$1 < 2 < 2^2$	$2^2.U_6(2)$
$C(16)$	$1 < 2^2$	$2^2.U_6(2).2$
$C(17)$	$1 < 2^2 < 2^{11}$	$2^{11}.L_3(4).2$
$C(18)$	$1 < 2 < 2^2 < 2^{11}$	$2^{11}.L_3(4)$
$C(19)$	$1 < 2^2 < 2^2 \times 2_+^{1+8}$	$(2^2 \times 2_+^{1+8}).U_4(2).2$
$C(20)$	$1 < 2 < 2^2 < 2^2 \times 2_+^{1+8}$	$(2^2 \times 2_+^{1+8}).U_4(2)$
$C(21)$	$1 < 2^2 < 2^{6+8}$	$2^{6+8}(3 \times A_5).2.2$
$C(22)$	$1 < 2 < 2^2 < 2^{6+8}$	$2^{6+8}(3 \times A_5).2$
$C(23)$	$1 < 2^2 \times 2_+^{1+8} < 2^{11}.2^4$	$2^{11}.2^4.3.S_5$
$C(24)$	$1 < 2 < 2^2 \times 2_+^{1+8} < 2^{11}.2^4$	$2^{11}.2^4.S_5$
$C(25)$	$1 < 2 < 2^2 < 2^{11} < 2^{11}.2^4$	$2^{11}.2^4.A_5$
$C(26)$	$1 < 2^2 < 2^2 \times 2_+^{1+8} < 2^{11}.2^4$	$2^{11}.2^4.S_5$
$C(27)$	$1 < 2 < 2^2 < 2^{11} < 2^{6+8}.2$	$2^{6+8}.2.A_5$
$C(28)$	$1 < 2^2 < 2^{6+8} < 2^{6+8}.2$	$2^{6+8}.2.S_5$
$C(29)$	$1 < 2 < 2^2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4$	$(2^2 \times 2_+^{1+8}).2.2^4.3.S_3$
$C(30)$	$1 < 2^2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4$	$(2^2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3)$
$C(31)$	$1 < 2^2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4 < 2^{11}.2^2.2^4$	$2^{11}.2^2.2^4.S_3$
$C(32)$	$1 < 2 < 2^2 < 2^{11} < 2^{11}.2^4 < S'$	$S'.3$

Table 7: Radical 2-chains of Fi_{23}

Theorem 6.1 *Let B be a 2-block of $G = \mathrm{Fi}_{23}$ with a positive defect. Then B satisfies the ordinary conjecture of Dade.*

PROOF: We may suppose $D(B)$ is noncyclic, so by Lemma 4.5 (c), $B = B_0$ or B_2. If $B = B_2$, then $D(B) \simeq D_8$ and by [13], B satisfies the ordinary conjecture of Dade. We may suppose $B = B_0$. We denote $\mathrm{k}(i, d) = \mathrm{k}(N(C(i)), B_0, d)$ for integers i, d.

First, we consider the 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 17$, so that $j \in \{15, 18, 20, 22, 25, 27, 29, 32\}$.

The subgroups $N(C(15)) \simeq 2^2.U_6(2)$ and $N(C(25)) \simeq 2^{11}.2^4.A_5$ have 139 and 143 irreducible characters, respectively, whose degrees are given in Tables C–1 and C–2. In addition, $N(C(15))$ has two blocks and the principal block contains 135 irreducible characters. Thus $\mathrm{k}(15, d)$ and $\mathrm{k}(25, d)$ are as in Table 8.

Defect d	17	16	15	14	13	12	11	10	7	otherwise
k(15, d)	16	8	4	32	21	30	16	4	4	0

Defect d	17	16	15	14	13	12	11	otherwise
k(25, d)	16	8	4	36	51	14	14	0

Table 8: Values of $\mathrm{k}(15, d)$ and $\mathrm{k}(25, d)$

The subgroups $N(C(27)) \simeq 2^{6+8}.2.A_5$ and $N(C(29)) \simeq (2^2 \times 2_+^{1+8}).2.2^4.3.S_3$ have 155 and 279 irreducible characters, respectively, whose degrees are given in Tables C–3 and C–4. Thus $\mathrm{k}(27, d)$ and $\mathrm{k}(29, d)$ are as in Table 9.

Defect d	17	16	15	14	13	12	11	otherwise
k(27, d)	16	8	24	28	67	4	8	0

Defect d	17	16	15	14	13	12	11	10	otherwise
k(29, d)	16	8	24	40	101	52	26	12	0

Table 9: Values of $\mathrm{k}(27, d)$ and $\mathrm{k}(29, d)$

If $\mathrm{k}(\mathrm{odd}_1, d) = \sum_{i \in \{15,25,27,29\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 10.

Defect d	17	16	15	14	13	12	11	10	7	otherwise
$\mathrm{k}(\mathrm{odd}_1, d)$	64	32	56	136	240	100	64	16	4	0

Table 10: Values of $\mathrm{k}(\mathrm{odd}_1, d)$

The subgroups $N(C(18)) \simeq 2^{11}.L_3(4)$ and $N(C(20)) \simeq (2^2 \times 2_+^{1+8}).U_4(2)$ have 89 and 201 irreducible characters, respectively, whose degrees are given in Tables C–5 and C–6. Thus $\mathrm{k}(18, d)$ and $\mathrm{k}(20, d)$ are as in Table 11.

Defect d	17	16	15	14	13	11	otherwise
k(18, d)	16	8	4	28	19	14	0

Defect d	17	16	15	14	13	12	11	10	7	otherwise
k(20, d)	16	8	4	40	53	48	16	12	4	0

Table 11: Values of $\mathrm{k}(18, d)$ and $\mathrm{k}(20, d)$

The subgroups $N(C(22)) \simeq 2^{6+8}(3 \times A_5).2$ and $N(C(32)) \simeq S'.3$ have 197 and 225 irreducible characters, respectively, whose degrees are given in Tables C–7 and C–8. Thus $\mathrm{k}(22, d)$ and $\mathrm{k}(32, d)$ are as in Table 12.

Defect d	17	16	15	14	13	12	11	10	otherwise
k(22, d)	16	8	24	32	69	34	10	4	0

Defect d	17	16	15	14	13	12	11	otherwise
k(32, d)	16	8	24	36	99	18	24	0

Table 12: Values of $\mathrm{k}(22, d)$ and $\mathrm{k}(32, d)$

If $\mathrm{k}(\mathrm{even}_1, d) = \sum_{i \in \{18,20,22,32\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 13.

Defect d	17	16	15	14	13	12	11	10	7	otherwise
$\mathrm{k}(\mathrm{even}_1, d)$	64	32	56	136	240	100	64	16	4	0

Table 13: Values of $\mathrm{k}(\mathrm{even}_1, d)$

It follows that

$$\sum_{i \in \{15,25,27,29\}} \mathrm{k}(N(C(i)), B_0, d) = \sum_{i \in \{18,20,22,32\}} \mathrm{k}(N(C(i)), B_0, d).$$

Next we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 18$ and $N(C(j)) \leq_{\mathrm{Fi}_{23}} 2.\mathrm{Fi}_{22}$, so that $j \in \{2, 3, 5, 7, 10, 11, 14, 24\}$.

The subgroups $N(C(3)) \simeq 2^{11}.M_{22}$ and $N(C(5)) \simeq (2^2 \times 2_+^{1+8}).U_4(2).2$ have 69 and 189 irreducible characters, respectively, whose degrees are given in Tables C–9 and C–10. Thus $\mathrm{k}(3, d)$ and $\mathrm{k}(5, d)$ are as in Table 14.

Defect d	18	17	16	15	14	13	12	11	otherwise
k(3, d)	16	12	2	8	12	14	2	3	0

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
k(5, d)	16	12	10	24	40	36	32	9	5	4	1	0

Table 14: Values of $\mathrm{k}(3, d)$ and $\mathrm{k}(5, d)$

The subgroups $N(C(7)) \simeq 2^{6+8}(S_3 \times A_6)$ and $N(C(11)) \simeq 2^{11}.2^2.2^4.S_3$ have 166 and 222 irreducible characters, respectively, whose degrees are given in Tables C–11 and C–12. Thus $\mathrm{k}(7, d)$ and $\mathrm{k}(11, d)$ are as in Table 15.

Defect d	18	17	16	15	14	13	12	11	10	otherwise
k(7, d)	16	12	18	14	33	43	18	8	4	0

Defect d	18	17	16	15	14	13	12	11	otherwise
k(11, d)	16	12	26	34	53	59	12	10	0

Table 15: Values of $\mathrm{k}(7, d)$ and $\mathrm{k}(11, d)$

If $\mathrm{k}(\mathrm{odd}_2, d) = \sum_{i \in \{3,5,7,11\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 16.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
$\mathrm{k}(\mathrm{odd}_2, d)$	64	48	56	80	138	152	64	30	9	4	1	0

Table 16: Values of $\mathrm{k}(\mathrm{odd}_2, d)$

The subgroups $N(C(2)) = 2.\mathrm{Fi}_{22}$ and $N(C(10)) \simeq 2^{6+8}.2.A_6$ have 114 and 132 irreducible characters, respectively, whose degrees are given in Tables C–13 and C–14. In addition, $N(C(2))$ has 3 blocks and the principal block contains 108 irreducible characters. Thus $\mathrm{k}(2, d)$ and $\mathrm{k}(10, d)$ are as in Table 17.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
k(2, d)	16	12	2	8	17	23	18	3	4	4	1	0

Defect d	18	17	16	15	14	13	12	11	otherwise
k(10, d)	16	12	18	14	28	34	2	8	0

Table 17: Values of $\mathrm{k}(2, d)$ and $\mathrm{k}(10, d)$

The subgroups $N(C(14)) \simeq (2^2 \times 2_+^{1+8})2.2^4.(S_3 \times S_3)$ and $N(C(24)) \simeq 2^{11}.2^4.S_5$ have 255 and 151 irreducible characters, respectively, whose degrees are given in Tables C–15 and C–16. Thus $\mathrm{k}(14, d)$ and $\mathrm{k}(24, d)$ are as in Table 18.

Defect d	18	17	16	15	14	13	12	11	10	otherwise
k(14, d)	16	12	26	30	56	64	32	14	5	0

Defect d	18	17	16	15	14	13	12	11	otherwise
k(24, d)	16	12	10	28	37	31	12	5	0

Table 18: Values of $\mathrm{k}(14, d)$ and $\mathrm{k}(24, d)$

If $\mathrm{k}(\mathrm{even}_2, d) = \sum_{i\in\{2,10,14,24\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 19.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
$\mathrm{k}(\mathrm{even}_2, d)$	64	48	56	80	138	152	64	30	9	4	1	0

Table 19: Values of $\mathrm{k}(\mathrm{even}_2, d)$

It follows that

$$\sum_{i\in\{3,5,7,11\}} \mathrm{k}(N(C(i)), B_0, d) = \sum_{i\in\{2,10,14,24\}} \mathrm{k}(N(C(i)), B_0, d).$$

Now we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 18$ and $N(C(j)) \leq_{\mathrm{Fi}_{23}} 2^2.U_6(2)$, so that $j \in \{16, 17, 19, 21, 26, 28, 30, 31\}$.

The subgroup $N(C(17)) \simeq 2^{11}.L_3(4).2$ has 97 irreducible characters, whose degrees are given in Table C–17. Thus $\mathrm{k}(17, d)$ is as in Table 20.

Defect d	18	17	16	15	14	13	12	11	otherwise
k(17, d)	16	12	10	20	19	7	8	5	0

Table 20: Values of $\mathrm{k}(17, d)$

The subgroup $N(C(19)) \simeq N(C(5)) \simeq (2^2 \times 2_+^{1+8}).U_4(2).2$ has 189 irreducible characters, whose degrees are given in Table C–10.

The subgroup $N(C(21)) \simeq 2^{6+8}(3 \times A_5).2.2$ has 199 irreducible characters, whose degrees are given in Table C–18. Thus $\mathrm{k}(21, d)$ is as in Table 21.

Defect d	18	17	16	15	14	13	12	11	10	otherwise
k(21, d)	16	12	26	30	36	48	24	6	1	0

Table 21: Values of $\mathrm{k}(21, d)$

The subgroup $N(C(31)) \simeq N(C(11)) \simeq 2^{11}.2^2.2^4.S_3$ has 222 irreducible characters, whose degrees are given in Table C–12.

If $\mathrm{k}(\mathrm{odd}_3, d) = \sum_{i\in\{17,19,21,31\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 22.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
$\mathrm{k}(\mathrm{odd}_3, d)$	64	48	72	108	148	150	76	30	6	4	1	0

Table 22: Values of $\mathrm{k}(\mathrm{odd}_3, d)$

The subgroup $N(C(16)) \simeq 2^2.U_6(2).2$ has 146 irreducible characters, whose degrees are given in Table C–19. In addition, $N(C(16))$ has two blocks and the principal block contains 141 irreducible characters. Thus $\mathrm{k}(16, d)$ is as in Table 23.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
k(16, d)	16	12	10	24	20	20	24	9	1	4	1	0

Table 23: Values of $\mathrm{k}(16, d)$

The subgroup $N(C(26)) \simeq N(C(24)) \simeq 2^{11}.2^4.S_5$ has 151 irreducible characters, whose degrees are given in Table C–16.

The subgroup $N(C(28)) \simeq 2^{6+8}.2.S_5$ has 160 irreducible characters, whose degrees are given in Table C–20. Thus $\mathrm{k}(28, d)$ is as in Table 24.

Defect d	18	17	16	15	14	13	12	11	otherwise
k(28, d)	16	12	26	26	35	35	8	2	0

Table 24: Values of $\mathrm{k}(28, d)$

The subgroup $N(C(30)) \simeq N(C(14)) \simeq (2^2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3)$ has 255 irreducible characters, whose degrees are given in Table C–15.

If $\mathrm{k}(\mathrm{even}_3, d) = \sum_{i \in \{16,26,28,30\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 25.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
$\mathrm{k}(\mathrm{even}_3, d)$	64	48	72	108	148	150	76	30	6	4	1	0

Table 25: Values of $\mathrm{k}(\mathrm{even}_3, d)$

It follows that

$$\sum_{i \in \{17,19,21,31\}} \mathrm{k}(N(C(i)), B_0, d) = \sum_{i \in \{16,26,28,30\}} \mathrm{k}(N(C(i)), B_0, d).$$

Finally, we consider the remaining chains $C(j)$, so that $j \in \{1, 4, 6, 8, 9, 12, 13, 23\}$.

If $j = 1$, then $N(C(j)) = \mathrm{Fi}_{23}$ and by Lemma 4.5 (c), $B = B_0$ has 89 irreducible characters. By MAGMA, $N(C(9)) \simeq 2^{6+8}.2.A_7$ has 105 irreducible characters, whose degrees are given in Table C–21. Thus the numbers $\mathrm{k}(1, d)$ and $\mathrm{k}(9, d)$ are as in Table 26.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
k(1, d)	16	20	10	8	7	12	2	7	2	4	1	0

Defect d	18	17	16	15	14	13	12	11	otherwise
k(9, d)	16	20	18	18	20	5	4	4	0

Table 26: Values of $\mathrm{k}(1, d)$ and $\mathrm{k}(9, d)$

The subgroups $N(C(13)) \simeq (2^2 \times 2_+^{1+8})2.2^4.3.(S_3 \times S_3)$ and $N(C(23)) \simeq 2^{11}.2^4.3.S_5$ have 254 and 146 irreducible characters, respectively, whose degrees are given in Tables C–22 and C–23. Thus $\mathrm{k}(13, d)$ and $\mathrm{k}(23, d)$ are as in Table 27.

Defect d	18	17	16	15	14	13	12	11	10	otherwise
k(13, d)	16	20	34	38	52	45	28	16	5	0

Defect d	18	17	16	15	14	13	12	11	otherwise
k(23, d)	16	20	10	32	29	26	10	3	0

Table 27: Values of $\mathrm{k}(13, d)$ and $\mathrm{k}(23, d)$

If $\mathrm{k}(\mathrm{odd}_4, d) = \sum_{i \in \{1,9,13,23\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 28.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
$\mathrm{k}(\mathrm{odd}_4, d)$	64	80	72	96	108	88	44	30	7	4	1	0

Table 28: Values of $\mathrm{k}(\mathrm{odd}_4, d)$

The subgroups $N(C(4)) = 2^{11}.M_{23}$ and $N(C(6)) = (2^2 \times 2_+^{1+8})(3 \times U_4(2)).2$ have 56 and 194 irreducible characters, respectively, whose degrees are given in Tables C–24 and C–25. Thus $\mathrm{k}(4, d)$ and $\mathrm{k}(6, d)$ are as in Table 29.

Defect d	18	17	16	15	14	13	11	otherwise
k(4, d)	16	20	2	8	4	1	5	0

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
k(6, d)	16	20	18	28	36	33	24	9	5	4	1	0

Table 29: Values of $\mathrm{k}(4, d)$ and $\mathrm{k}(6, d)$

The subgroups $N(C(8)) \simeq 2^{6+8}(S_3 \times A_7)$ and $N(C(12)) \simeq 2^{11}.2^2.2^4.F_{3^2}^2$ have 133 and 211 irreducible characters, respectively, whose degrees are given in Tables C–26 and C–27. Thus $\mathrm{k}(8, d)$ and $\mathrm{k}(12, d)$ are as in Table 30.

Defect d	18	17	16	15	14	13	12	11	10	otherwise
k(8, d)	16	20	26	18	23	16	6	6	2	0

Defect d	18	17	16	15	14	13	12	11	otherwise
k(12, d)	16	20	26	42	45	38	14	10	0

Table 30: Values of $\mathrm{k}(8, d)$ and $\mathrm{k}(12, d)$

If $\mathrm{k}(\mathrm{even}_4, d) = \sum_{i\in\{4,6,8,12\}} \mathrm{k}(N(C(i)), B_0, d)$, then the values are recorded in Table 31.

Defect d	18	17	16	15	14	13	12	11	10	8	7	otherwise
$\mathrm{k}(\mathrm{even}_4, d)$	64	80	72	96	108	88	44	30	7	4	1	0

Table 31: Values of $\mathrm{k}(\mathrm{even}_4, d)$

It follows that

$$\sum_{i\in\{1,9,13,23\}} \mathrm{k}(N(C(i)), B_0, d) = \sum_{i\in\{4,6,8,12\}} \mathrm{k}(N(C(i)), B_0, d).$$

This completes the proof. □

Theorem 6.2 *Let B be a p-block of $G = \mathrm{Fi}_{23}$ with a positive defect. If p is odd, then B satisfies the ordinary conjecture of Dade.*

The proof of Theorem 6.2 is similar to that of Theorem 6.1 and is in Appendix B.

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Department of Mathematics
University of Auckland
Private Bag 92019
Auckland
New Zealand

Email: an@math.auckland.ac.nz obrien@math.auckland.ac.nz

A Proof of Lemma 5.3

PROOF: (b) Let $C \in \mathcal{R}(G)$ be given by (2.1) with $P_1 \in \Phi(G, 2)$.

Case (1). Let $\mathcal{M}$ be the subfamily of $\mathcal{R} = \mathcal{R}(G)$ consisting of chains C such that $P_1 =_G (2^2)^* = O_2(M_6)$. Then $\mathcal{M}$ is G-invariant, and we may suppose $P_1 = (2^2)^*$ and $P_2 \in \Phi(M_6, 2)$ given by (4.12) when $|C| \geq 2$. Write $P_1 = X_1 \times Y_1$, where $X_1 = (2^2)^*$ and $Y_1 = 1 \leq S_6(2)$. Since $M_6 = S_4 \times S_6(2)$, it follows that $P_2 = X_2 \times Y_2$, where $X_2 \in \{(2^2)^*, D_8\}$ and $Y_2 \in \Phi(S_6(2), 2)$. Moreover, the i-th subgroup P_i of $C \in \mathcal{M}$ decomposes as

$$P_i = X_i \times Y_i,$$

where $X_i = (2^2)^*$ or D_8 and $Y_i \leq S_6(2)$. It is clear that if $X_i = D_8$, then $X_j = D_8$ for all $j \geq i$. If $n = |C|$ and $X_n =_{S_4} (2^2)^*$, set $\ell(C) = 0$, otherwise set $\ell = \ell(C)$ to be the smallest integer such that $X_\ell = D_8$, so that $\ell(C) = 0$ or $\ell(C) \geq 2$ for $C \in \mathcal{M}$. Denote by $\mathcal{M}_+$ the subfamily of $\mathcal{M}$ consisting of chains C such that $\ell(C) \geq 2$ and either $|C| - \ell(C) \geq 1$ or $|C| = \ell(C)$ but $Y_{\ell-1} \neq Y_\ell$.

Suppose $C \in \mathcal{M}_+$ given by (2.1) and $\ell = \ell(C)$. Then

$$C : 1 < \ldots < P_{\ell-1} = X_{\ell-1} \times Y_{\ell-1} < P_\ell = X_\ell \times Y_\ell < \ldots < P_n$$

and $Y_{\ell-1} \leq Y_\ell$, where $X_{\ell-1} = (2^2)^*$ and $X_\ell = D_8$. If $Y_{\ell-1} \neq Y_\ell$, then define

$$\varphi(C) : 1 < \ldots < P_{\ell-1} < D_8 \times Y_{\ell-1} < P_\ell < \ldots < P_n.$$

If $Y_{\ell-1} = Y_\ell$, then $|C| - \ell(C) \geq 1$ and $Y_{\ell+1} \neq Y_\ell$. Since $\cap_{i=1}^{\ell} N_{S_6(2)}(Y_i) = \cap_{i=1}^{\ell-1} N_{S_6(2)}(Y_i)$, it follows that $Y_{\ell+1}$ is a radical 2-subgroup of $\cap_{i=1}^{\ell-1} N_{S_6(2)}(Y_i)$, so that $P_{\ell+1}$ is a radical subgroup of $\cap_{i=1}^{\ell-1} N_G(P_i)$. We define

$$\varphi(C) : 1 < \ldots < P_{\ell-1} < P_{\ell+1} < \ldots < P_n.$$

Thus φ is an involutive mapping of $\mathcal{M}_+$, $|\varphi(C)| = |C| \mp 1$ and $N(C) = N(\varphi(C))$. Thus

$$\sum_{C \in \mathcal{M}_+} (-1)^{|C|} \mathrm{k}(N(C), B, d) = 0$$

for all $B \in \mathrm{Blk}(G)$ and integers d, so we may suppose $C \in \mathcal{M} \backslash \mathcal{M}_+$.

Given $i = 1$ or 0, let $\mathcal{M}_i$ be the subfamily of $\mathcal{M} \backslash \mathcal{M}_+$ consisting of chains C such that $\ell(C) = |C|$ or 0 according as $i = 1$ or 0. If $C \in \mathcal{M}_1$, then

$$C : 1 < \ldots < P_{n-1} = (2^2)^* \times Y_{n-1} < P_n = D_8 \times Y_n$$

and $Y_{n-1} = Y_n$. We define $\varphi(C) : 1 < \ldots < P_{n-1}$, so that $\varphi(C) \in \mathcal{M}_0$. If $C \in \mathcal{M}_0$, then

$$C : 1 < \ldots < P_n = (2^2)^* \times Y_n$$

and define $\varphi(C)$ to be the chain $1 < \ldots < P_n < D_8 \times Y_n$, so that $\varphi(C) \in \mathcal{M}_1$. Since $\varphi(\varphi(C)) = C$ for each $C \in \mathcal{M}_0 \cup \mathcal{M}_1$, it follows that φ induces a bijection from $\mathcal{M}_1$ to $\mathcal{M}_0$. Suppose $C \in \mathcal{M}_1$ with $|C| = n$. Then $N(C) = D_8 \times H$ and $N(\varphi(C)) = S_4 \times H$, where $H = \cap_{i=1}^{n} N_{S_6(2)}(Y_i)$. Now D_8 has 4 linear characters and one irreducible character of degree 2, and S_4 has 2 linear characters, 2 irreducible of degree 3 and one of degree 2, and moreover, both D_8 and S_4 have exactly one block, the principal block $B_0(D_8)$ and $B_0(S_4)$, respectively. Thus there is a 2-defect preserved bijection ψ from $\mathrm{Irr}(D_8)$ to $\mathrm{Irr}(S_4)$. If ξ is a character of $\mathrm{Irr}(N(C), B, d)$, then $\xi = \xi_1 \times \xi_2$, where $\xi_1 \in \mathrm{Irr}(D_8)$ and $\xi_2 \in \mathrm{Irr}(H)$. In addition, the block $B(\xi)$ of $N(C)$ containing ξ has the form $B(\xi) = B_0(D_8) \times b_H$ for some $b_H \in \mathrm{Blk}(H)$. If $\Psi(\xi) = \psi(\xi_1) \times \xi_2$, then $B(\Psi(\xi)) = B_0(S_4) \times b_H$ and both $\Psi(\xi)$ and ξ have the same defect d. Since $B(\xi)^G = B$ and $B(\xi)^{N(\varphi(C))} = B(\Psi(\xi))$, it follows that $B(\Psi(\xi))^G = B$, so that

$$\Psi(\xi) \in \mathrm{Irr}(N(\varphi(C)), B, d).$$

Since ψ is a defect preserved bijection, $\mathrm{Irr}(B_0(D_8)) = \mathrm{Irr}(D_8)$ and $\mathrm{Irr}(B_0(S_4)) = \mathrm{Irr}(S_4)$, it follows that Ψ is a bijection between $\mathrm{Irr}(N(C), B, d)$ and $\mathrm{Irr}(N(\varphi(C)), B, d)$, so that $\mathrm{k}(N(C), B, d) = \mathrm{k}(N(\varphi(C)), B, d)$ and

$$\sum_{C \in \mathcal{M}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = 0.$$

We may suppose $C \notin \mathcal{M}$.

Case (2). Suppose $P_1 = 2 = O_2(M_1)$, and so we may assume that $P_2 \in \Phi(M_1, 2)$ when $|C| \geq 2$. Let

$$\Omega = \{2^2, 2^{11}, 2^2 \times 2_+^{1+8}, 2^{6+8}, 2^{11}.2^4, 2^{6+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^4\}$$

be a subset of $\Phi(M_1, 2)$ and $2 \neq R \in \Phi(M_1, 2) \backslash \Omega$, so that $N_{M_1}(R) = N(R)$. Let $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ be the subfamilies of $\mathcal{R}$ defined by (5.3) with 3_+^{1+8} replaced by 2. Then for $C' \in \mathcal{M}^+(R)$ given by (5.2),

$$g(C') : 1 < 2 < P'_1 = R < \ldots < P'_m$$

is a chain of $\mathcal{M}^0(R)$ and $N(C') = N(g(C'))$, so that (5.5) holds. We may suppose

$$C \notin \bigcup_{R \in \Phi(M_1, 2) \backslash \Omega} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R))$$

and in particular, $P_1 \notin_G \Phi(M_1, 2)) \backslash \Omega$. Moreover, if $P_1 = 2$ and $|C| \geq 2$, then $P_2 \in_G \Omega$.

Case (2.1). By Magma, $N_{M_1}(2^{11}) = 2^{11}.M_{22}$ and we may take $\Phi(N_{M_1}(2^{11}), 2)$ as a subset of $\Phi(M_1, 2)$ such that

$$\Phi(2^{11}.M_{22}, 2) = \{2^{11}, 2^{11}.2^3, 2^{11}.2^4, 2^{6+8}.2, 2^{11}.2^2.2^3, 2^{6+8}.2^3, 2^{11}.2^2.2^4, S\}$$

and moreover, if $R \in \mathcal{X} = \{2^{11}.2^4, 2^{6+8}.2, 2^{11}.2^2.2^4\} \subseteq \Omega$, then $N_{M_1}(R) \leq N_{M_1}(2^{11})$; and if $2^{11} \neq R \in \Phi(N_{M_1}(2^{11}), 2) \backslash \mathcal{X}$, then $N(R) = N_{M_1}(R)$.

Given $Q \in \mathcal{X}$, let $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ be the subfamilies of $\mathcal{R}$ defined by (5.6) with 3 replaced by 2 and 3^6 by 2^{11}. Then a similar proof shows that there is a bijection g between $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ satisfying $N(C') = N(g(C'))$. Thus we may suppose (5.7) holds with Ω replaced by $\mathcal{X}$.

If $P_1 = 2$ and $P_2 \in \Omega$, we may suppose $P_2 \notin_G \mathcal{X}$ and moreover, if $P_2 = 2^{11}$, then either $C =_G C(3)$ or $P_3 \in \Phi(N_{M_1}(2^{11}), 2) \backslash \mathcal{X}$.

Case (2.2). By MAGMA, $N_{M_1}(2^2 \times 2_+^{1+8}) = (2^2 \times 2_+^{1+8}).U_4(2).2$ and we may take $\Phi(N_{M_1}(2^2 \times 2_+^{1+8}), 2)$ as a subset of $\Phi(M_1, 2)$ such that

$$\begin{aligned}\Phi((2^2 \times 2_+^{1+8}).U_4(2).2, 2) = \quad & \{2^2 \times 2_+^{1+8}, (2^2 \times 2_+^{1+8}).2, 2^{11}.2^4, \\ & (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^3, 2^{6+8}.D_8, 2^{11}.2^2.2^4, S\}\end{aligned}$$

and moreover, if $R \in \mathcal{Y} = \{2^{11}.2^4, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^4\} \subseteq \Omega$, then $N_{M_1}(R) \leq N_{M_1}(2^2 \times 2_+^{1+8})$; and if $2^2 \times 2_+^{1+8} \neq R \in \Phi(N_{M_1}(2^2 \times 2_+^{1+8}), 2)\backslash\mathcal{Y}$, then $N_G(R) = N_{M_1}(R)$.

Let $\mathcal{L}^+((2^2 \times 2_+^{1+8}).2.2^4)$ and $\mathcal{L}^0((2^2 \times 2_+^{1+8}).2.2^4)$ be the subfamilies of $\mathcal{R}$ defined by (5.6) with 3 replaced by 2, 3^6 by $2^2 \times 2_+^{1+8}$ and Q by $(2^2 \times 2_+^{1+8}).2.2^4$. A similar proof shows that we may suppose

$$C \notin (\mathcal{L}^+((2^2 \times 2_+^{1+8}).2.2^4) \cup \mathcal{L}^0((2^2 \times 2_+^{1+8}).2.2^4)),$$

so that if $P_1 = 2$, then $P_2 \neq_G (2^2 \times 2_+^{1+8}).2.2^4$, and moreover, if $P_2 = 2^2 \times 2_+^{1+8}$, then either $C =_G C(5)$ or $P_3 \in \Phi(N_{M_1}(2^2 \times 2_+^{1+8}), 2)\backslash\{(2^2 \times 2_+^{1+8}).2.2^4\}$.

Case (2.3). By MAGMA, the normalizer $N_{M_1}(2^{6+8}) = 2^{6+8}.(S_3 \times A_6)$ and we may suppose $\Phi(N_{M_1}(2^{6+8}), 2)$ is a subset of $\Phi(M_1, 2)$ such that

$$\Phi(N_{M_1}(2^{6+8}), 2) = \{2^{6+8}, 2^{6+8}.2, 2^{6+8}.2^2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^4, 2^{6+8}.2^3, 2^{6+8}.D_8, S\}$$

and moreover, if $R \in \mathcal{Z} = \{2^{6+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^4\} \subseteq \Omega$, then $N_{M_1}(R) \leq N_{M_1}(2^{6+8})$; and if $2^{6+8} \neq R \in \Phi(N_{M_1}(2^{6+8}), 2)\backslash\mathcal{Z}$, then $N_G(R) = N_{M_1}(R)$.

If $P_1 = 2$ and $P_2 = 2^{6+8}$, then either $C =_G C(7)$ or $P_3 \in \Phi(N_{M_1}(2^{6+8}), 2)$.

Case (2.4). By MAGMA, $N_{M_1}(2^2) = 2^2.U_6(2)$ and we may take

$$\Phi(2^2.U_6(2), 2) = \{2^2, 2^2 \times 2_+^{1+8}, 2^{11}, 2^{6+8}, 2^{11}.2^4, 2^{6+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, S'\},$$

where $S' \in \mathrm{Syl}_2(2^2.U_6(2))$. Moreover,

$$N_{2^2.U_6(2)}(R)/R = \begin{cases} U_4(2) & \text{if } R = 2^2 \times 2_+^{1+8}, \\ L_3(4) & \text{if } R = 2^{11}, \\ (3 \times A_5).2 & \text{if } R = 2^{6+8}, \\ A_5 & \text{if } R = 2^{11}.2^4 \text{ or } 2^{6+8}.2, \\ S_3 \times S_3 & \text{if } (2^2 \times 2_+^{1+8}).2.2^4, \\ 3 & \text{if } R = S'. \end{cases}$$

If $P_1 = 2$ and $P_2 = 2^2$, then either $C =_G C(15)$ or $P_3 \in \Phi(2^2.U_6(2), 2)$.

Case (3). Suppose $P_1 = 2^{11} = O_2(M_3)$, so that we may take $P_2 \in_G \Phi(M_3, 2)$ which is given in the proof (4) of Lemma 4.4. Let $\mathcal{X}^* = \{2^{11}.2^3, 2^{11}.2^2.2^3, 2^{6+8}.2^3, S\}$ be a subset of $\Phi(M_3, 2)$. We may suppose $\mathcal{X}^*$ is a subset of $\Phi(2^{11}.M_{22}, 2)$ given by Case (2.1) above, so that $\mathcal{X}^* = \Phi(2^{11}.M_{22}, 2)\backslash(\mathcal{X} \cup \{2^{11}, 2^{11}.2^4\})$ and $N_{M_3}(R) = N_{2^{11}.M_{22}}(R) = N(R)$ for each $R \in \mathcal{X}^*$. Given $W \in \mathcal{X}^*$, let $\mathcal{G}^+(W)$ and $\mathcal{G}^0(W)$ be the subfamilies of $\mathcal{R}$ such that

$$\begin{aligned}\mathcal{G}^+(W)/G &= \{C' \in \mathcal{R}/G : P_1' = 2^{11}, P_2' = W\}, \\ \mathcal{G}^0(W)/G &= \{C' \in \mathcal{R}/G : P_1' = 2, P_2' = 2^{11}, P_3' = W\}. \qquad (5.8)\end{aligned}$$

If $C' \in \mathcal{G}^+(W)$ is given by (5.2), then

$$g(C') : 1 < 2 < P_1' < \ldots < P_m'$$

is a chain of $\mathcal{G}^0(W)$ and $N(C') = N(g(C'))$. Moreover, g is a bijection between $\mathcal{G}^+(W)$ and $\mathcal{G}^0(W)$, so we may suppose

$$C \notin \bigcup_{W \in \mathcal{X}^*} (\mathcal{G}^+(W) \cup \mathcal{G}^0(W)). \tag{5.9}$$

In particular, if $P_1 = 2$ and $P_2 =_G 2^{11}$, then $C =_G C(3)$; if $P_1 = 2^{11}$ and $|C| \geq 2$, then $P_2 \in_G \Phi(M_3, 2) \backslash \mathcal{X}^*$.

Suppose $R \in \Phi(M_3, 2) \backslash (\mathcal{X}^* \cup \{2^{11}\}) = \{2^{11}.2^4, 2^{6+8}.2, 2^{11}.2^2.2^4\}$. Let $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ be the subfamilies of $\mathcal{R}$ defined by (5.3) with 3_+^{1+8} replaced by 2^{11}. A similar proof shows that there is a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$, and we may suppose (5.5) holds, so that

$$C \notin \bigcup_{R \in \Phi(M_3,2) \backslash (\mathcal{X}^* \cup \{2^{11}\})} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)).$$

It follows that if $P_1 =_G 2^{11}$, then $C =_G C(4)$.

Case (4). Suppose $P_1 = 2^2 \times 2_+^{1+8} = O_2(M_4)$, so that we may take $P_2 \in_G \Phi(M_4, 2)$ which is given in the proof (3) of Lemma 4.4. Let

$$\mathcal{Y}^* = \{(2^2 \times 2_+^{1+8}).2, 2^{11}.2^2.2^3, 2^{6+8}.D_8, S\}$$

be a subset of $\Phi(M_4, 2)$. We may suppose $\mathcal{Y}^*$ is a subset of $\Phi((2^2 \times 2_+^{1+8}).U_4(2).2, 2)$ given by Case (2.2) above. Thus $N_{M_4}(R) = N_{(2^2 \times 2_+^{1+8}).U_4(2).2}(R) = N(R)$ for each $R \in \mathcal{Y}^*$. If $W \in \mathcal{Y}^*$, let $\mathcal{G}^+(W)$ and $\mathcal{G}^0(W)$ be the subfamilies of $\mathcal{R}$ given by (5.8) with 2^{11} replaced by $2^2 \times 2_+^{1+8}$. A similar proof shows that we may suppose (5.9) holds with $\mathcal{X}^*$ replaced by $\mathcal{Y}^*$.

In particular, if $P_1 = 2$ and $P_2 =_G 2^2 \times 2_+^{1+8}$, then either $C =_G C(5)$ or $P_3 =_G 2^{11}.2^4$ or $2^{11}.2^2.2^4$; if $P_1 = 2^2 \times 2_+^{1+8}$ and $|C| \geq 2$, then $P_2 \in_G \Phi(M_4, 2) \backslash (\mathcal{Y}^* \cup \{2^2 \times 2_+^{1+8}\}) = \{2^{11}.2^4, 2^{11}.2^2.2^4, (2^2 \times 2_+^{1+8}).2.2^4\}$.

Let $\mathcal{M}^+((2^2 \times 2_+^{1+8}).2.2^4)$ and $\mathcal{M}^0((2^2 \times 2_+^{1+8}).2.2^4)$ be the subfamilies of $\mathcal{R}$ defined by (5.3) with 3_+^{1+8} replaced by $2^2 \times 2_+^{1+8}$ and R by $(2^2 \times 2_+^{1+8}).2.2^4$. We may suppose (5.5) holds, so that

$$C \notin (\mathcal{M}^+((2^2 \times 2_+^{1+8}).2.2^4) \cup \mathcal{M}^0((2^2 \times 2_+^{1+8}).2.2^4)).$$

In particular, we may suppose $P_1 \neq_G (2^2 \times 2_+^{1+8}).2.2^4$; and if $P_1 =_G 2^2 \times 2_+^{1+8}$, then either $C =_G C(6)$ or $P_2 \in_G \{2^{11}.2^4, 2^{11}.2^2.2^4\}$.

Case (4.1). By Magma, $N_{M_4}(2^{11}.2^4) = 2^{11}.2^4.3.S_5 = N(2^{11}.2^4)$ and we may take

$$\Phi(2^{11}.2^4.3.S_5, 2) = \{2^{11}.2^4, 2^{11}.2^2.2^3, 2^{11}.2^2.2^4, S\} \subseteq \Phi(G, 2)$$

and $N_{N(2^{11}.2^4)}(R) = N(R)$ for each $R \in \Phi(2^{11}.2^4.3.S_5, 2)$. Let $\mathcal{L}^+(2^{11}.2^2.2^4)$ and $\mathcal{L}^0(2^{11}.2^2.2^4)$ be the subfamilies of $\mathcal{R}$ defined by (5.6) with 3 replaced by $2^2 \times 2_+^{1+8}$, 3^6 by $2^{11}.2^4$ and Q by $2^{11}.2^2.2^4$. Then we may suppose

$$C \notin (\mathcal{L}^+(2^{11}.2^2.2^4) \cup \mathcal{L}^0(2^{11}.2^2.2^4)),$$

so that if $P_1 = 2^2 \times 2_+^{1+8}$, then $P_2 \neq_G 2^{11}.2^2.2^4$, and moreover, if $P_2 =_G 2^{11}.2^4$, then $P_3 \in_G \Phi(N(2^{11}.2^4), 2) \backslash \{2^{11}.2^4, 2^{11}.2^2.2^4\}$.

Case (4.2). By Magma, $N_{N_{M_1}(2^2 \times 2_+^{1+8})}(2^{11}.2^4) = 2^{11}.2^4.S_5$ and we may take

$$\Phi(2^{11}.2^4.S_5, 2) = \Phi(2^{11}.2^4.3.S_5, 2).$$

Thus $N_{2^{11}.2^4.S_5}(2^{11}.2^2.2^4) = N_{N_{M_1}(2^2 \times 2_+^{1+8})}(2^{11}.2^2.2^4)$ and $N_{2^{11}.2^4.S_5}(R) = N(R)$ for $R \in \{2^{11}.2^2.2^3, S\}$. Suppose $D \in \{2^{11}.2^2.2^3, S\}$. Let $\mathcal{T}^+(D)$ and $\mathcal{T}^0(D)$ be the subfamilies of $\mathcal{R}$ such that

$$\begin{aligned} \mathcal{T}^+(D)/G &= \{C' \in \mathcal{R}/G : P_1' = 2^2 \times 2_+^{1+8}, P_2' = 2^{11}.2^4, P_3' = D\}, \\ \mathcal{T}^0(D)/G &= \{C' \in \mathcal{R}/G : P_1' = 2, P_2' = 2^2 \times 2_+^{1+8}, P_3' = 2^{11}.2^4, P_4' = D\}. \end{aligned} \tag{5.10}$$

A similar proof to that of (5.7) shows that we may suppose

$$C \notin \bigcup_{D \in \{2^{11}.2^2.2^3, S\}} (\mathcal{T}^+(D) \cup \mathcal{T}^0(D)). \tag{5.11}$$

If $P = 2^{11}.2^2.2^4$, then define

$$\begin{aligned} \mathcal{H}^+(P)/G &= \{C' \in \mathcal{R}/G : P_1' = 2, P_2' = 2^2 \times 2_+^{1+8}, P_3' = P\}, \\ \mathcal{H}^0(P)/G &= \{C' \in \mathcal{R}/G : P_1' = 2, P_2' = 2^2 \times 2_+^{1+8}, P_3' = 2^{11}.2^4, P_4' = P\}. \end{aligned} \tag{5.12}$$

Then a similar proof shows that we may suppose

$$C \notin (\mathcal{H}^+(P) \cup \mathcal{H}^0(P)). \tag{5.13}$$

Thus if $P_1 =_G 2$ and $P_2 = 2^2 \times 2_+^{1+8}$, then $C \in \{C(5), C(24)\}$; if $P_1 = 2^2 \times 2_+^{1+8}$, then $C \in \{C(6), C(23)\}$.

Case (5). Suppose $P_1 = 2^{6+8} = O_2(M_5)$, so that we may take $P_2 \in_G \Phi(M_5, 2)$ which is given in the proof (2) of Lemma 4.4. Let $\mathcal{Z}^* = \{2^{6+8}.2^2, 2^{6+8}.2^3, 2^{6+8}.D_8, S\}$ be a subset of $\Phi(M_5, 2)$. We may suppose $\mathcal{Z}^*$ is a subset of $\Phi(2^{6+8}.(S_3 \times A_6), 2)$ given by Case (2.3) above. Thus $N_{M_5}(R) = N_{2^{6+8}.(S_3 \times A_6)}(R) = N(R)$ for each $R \in \mathcal{Z}^*$. If $W \in \mathcal{Z}^*$, let $\mathcal{G}^+(W)$ and $\mathcal{G}^0(W)$ be the subfamilies of $\mathcal{R}$ given by (5.8) with 2^{11} replaced by 2^{6+8}. A similar proof shows that we may suppose (5.9) holds with $\mathcal{X}^*$ replaced by $\mathcal{Z}^*$.

In particular, if $P_1 = 2$ and $P_2 =_G 2^{6+8}$, then either $C =_G C(7)$ or $P_3 \in_G \mathcal{Z}$ given in the proof Case (2.3); if $P_1 = 2^{6+8}$ and $|C| \geq 2$, then $P_2 \in_G \mathcal{Z}$, where $\mathcal{Z}$ is identified as a subset of $\Phi(M_5, 2)$.

Case (5.1). By Magma, $N_{M_5}(2^{6+8}.2) = 2^{6+8}.2.A_7 = N(2^{6+8}.2)$ and we may take

$$\Phi(2^{6+8}.2.A_7, 2) = \{2^{6+8}.2, 2^{6+8}.2^3, 2^{11}.2^2.2^4, S\} \subseteq \Phi(G, 2)$$

and $N_{2^{6+8}.2.A_7}(R) = N(R)$ for each $R \in \Phi(2^{6+8}.2.A_7, 2)$. Let $\mathcal{L}^+(2^{11}.2^2.2^4)$ and $\mathcal{L}^0(2^{11}.2^2.2^4)$ be the subfamilies of $\mathcal{R}$ defined by (5.6) with 3 replaced by 2^{6+8}, 3^6 by $2^{6+8}.2$ and Q by $2^{11}.2^2.2^4$. Then we may suppose

$$C \notin (\mathcal{L}^+(2^{11}.2^2.2^4) \cup \mathcal{L}^0(2^{11}.2^2.2^4)),$$

so that if $P_1 = 2^{6+8}$, then $P_2 \neq_G 2^{11}.2^2.2^4$, and moreover, if $P_2 =_G 2^{6+8}.2$, then $P_3 \in_G \Phi(2^{6+8}.2.A_7, 2)\backslash\{2^{6+8}.2, 2^{11}.2^2.2^4\}$.

Case (5.2). By MAGMA, $N_{M_5}((2^2 \times 2_+^{1+8}).2.2^4) = (2^2 \times 2_+^{1+8}).2.2^4.3.(S_3 \times S_3) = N((2^2 \times 2_+^{1+8}).2.2^4)$ and we may take

$$\Phi(N((2^2 \times 2_+^{1+8}).2.2^4), 2) = \{(2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.D_8, 2^{11}.2^2.2^4, S\} \subseteq \Phi(G, 2)$$

and $N_{N((2^2\times 2_+^{1+8}).2.2^4)}(R) = N(R)$ for each $R \in \Phi(N((2^2 \times 2_+^{1+8}).2.2^4), 2)$.

Case (5.3). By MAGMA, $N_{N_{M_1}(2^{6+8})}(2^{6+8}.2) = 2^{6+8}.2.A_6$ and we may take

$$\Phi(2^{6+8}.2.A_6, 2) = \Phi(2^{6+8}.2.A_7, 2).$$

In addition, $N_{2^{6+8}.2.A_6}(R) = N(R)$ for $R \in \{2^{6+8}.2^3, S\}$ and $N_{2^{6+8}.2.A_6}(2^{11}.2^2.2^4) = N_{N_{M_1}(2^{6+8})}(2^{11}.2^2.2^4)$. Suppose $D \in \{2^{6+8}.2^3, S\}$. Let $\mathcal{T}^+(D)$ and $\mathcal{T}^0(D)$ be the subfamilies of $\mathcal{R}$ defined by (5.10) with $2^2 \times 2_+^{1+8}$ replaced by 2^{6+8} and $2^{11}.2^4$ by $2^{6+8}.2$. Then we may suppose (5.11) holds with $D \in \{2^{6+8}.2^3, S\}$. Let $\mathcal{H}^+(2^{11}.2^2.2^4)$ and $\mathcal{H}^0(2^{11}.2^2.2^4)$ be the subfamilies defined by (5.12) with $2^2 \times 2_+^{1+8}$ replaced by 2^{6+8} and $2^{11}.2^4$ by $2^{6+8}.2$. Then we may suppose

$$C \notin (\mathcal{H}^+(2^{11}.2^2.2^4) \cup \mathcal{H}^0(2^{11}.2^2.2^4)).$$

Case (5.4). By MAGMA, $N_{N_{M_1}(2^{6+8})}((2^2\times 2_+^{1+8}).2.2^4) = (2^2\times 2_+^{1+8}).2.2^4.(S_3\times S_3)$ and we may take $\Phi((2^2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3), 2) = \Phi(N((2^2 \times 2_+^{1+8}).2.2^4), 2)$. In addition, $N_{(2^2\times 2_+^{1+8}).2.2^4.(S_3\times S_3)}(R) = N(R)$ for $R \in \{2^{6+8}.D_8, S\}$ and

$$N_{(2^2\times 2_+^{1+8}).2.2^4.(S_3\times S_3)}(2^{11}.2^2.2^4) = 2^{11}.2^2.2^4.S_3.$$

Suppose $D \in \{2^{6+8}.D_8, S\}$. Let $\mathcal{T}^+(D)$ and $\mathcal{T}^0(D)$ be the subfamilies of $\mathcal{R}$ defined by (5.10) with $2^2 \times 2_+^{1+8}$ replaced by 2^{6+8} and $2^{11}.2^4$ by $(2^2 \times 2_+^{1+8}).2.2^4$. Then we may suppose (5.11) holds with $D \in \{2^{6+8}.D_8, S\}$.

Let $C' : 1 < 2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4 < 2^{11}.2^2.2^4 < S$ and

$$g(C') : 1 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4 < 2^{11}.2^2.2^4 < S.$$

Then $N(C') = N(g(C')) = N(S)$, so that (5.5) holds and we may suppose $C \neq_G C', g(C')$.

It follows that if $P_1 =_G 2^{6+8}$, then $C \in_G \{C(8), C(9), C(12), C(13)\}$; if $P_1 =_G 2$ and $P_2 =_G 2^{6+8}$, then $C \in_G \{C(7), C(10), C(11), C(14)\}$.

Case (6). Suppose $P_1 = 2^2 = O_2(M_2)$, so that we may take $P_2 \in_G \Phi(M_2, 2)$ which is given in the proof (5) of Lemma 4.4. Let $\mathcal{M}^+(D_8)$ and $\mathcal{M}^0(D_8)$ be the subfamilies of $\mathcal{R}$ defined by (5.3) with 3_+^{1+8} replaced by 2^2 and R by D_8. Then we may suppose $C \notin (\mathcal{M}^+(D_8) \cup \mathcal{M}^0(D_8))$. In particular, $P_1 \neq_G D_8$ and if $P_1 =_G 2$, then $P_2 \neq_G D_8$.

Case (6.1). As shown in the proof (5) of Lemma 4.4, $N_{M_2}(2^{11}) = 2^{11}.L_3(4).2$ and we may take

$$\Phi(2^{11}.L_3(4).2, 2) = \{2^{11}, 2^{11}.2, 2^{11}.2^4, 2^{6+8}.2, 2^{6+8}.2^2, 2^{11}.2^2.2^3, 2^{11}.2^2.2^4, S\}$$

and $N_{N_{M_2}(2^{11})}(R) = N_{M_2}(R)$ for each $R \in \Phi(2^{11}.L_3(4).2, 2)$. Moreover, we may suppose $\Phi(2^{11}.L_3(4).2, 2) \subseteq \Phi(M_2, 2)$.

Given $Q \in \Phi(2^{11}.L_3(4).2, 2)\backslash\{2^{11}\}$, let $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ be the subfamilies of $\mathcal{R}$ defined by (5.6) with 3 replaced by 2^2, 3^6 by 2^{11}. Then we may suppose (5.7) holds with Ω replaced by $\Phi(2^{11}.L_3(4).2, 2)\backslash\{2^{11}\}$. Thus if $P_1 =_G 2^2$ and $P_2 =_G 2^{11}$, then $C =_G C(17)$; if $P_1 =_G 2^2$, then $P_2 \in_G \Phi(M_2, 2)\backslash\Phi(2^{11}.L_3(4).2, 2)$.

Case (6.2). As shown in the proof (5) of Lemma 4.4,

$$N_{M_2}(2^2 \times 2_+^{1+8}) = (2^2 \times 2_+^{1+8}).U_4(2).2$$

and we may take

$$\begin{aligned}\Phi((2^2 \times 2_+^{1+8}).U_4(2).2, 2) \;=\; & \{2^2 \times 2_+^{1+8}, (2^2 \times 2_+^{1+8}).2, 2^{11}.2^4, \\ & (2^2 \times 2_+^{1+8}).2.2^4, 2^{11}.2^2.2^3, 2^{6+8}.D_8, 2^{11}.2^2.2^4, S\}\end{aligned}$$

and $N_{N_{M_2}(2^2\times 2_+^{1+8})}(R) = N_{M_2}(R)$ for each $R \in \Phi((2^2 \times 2_+^{1+8}).U_4(2).2, 2)$. We may suppose $\Phi((2^2 \times 2_+^{1+8}).U_4(2).2, 2) \subseteq \Phi(M_2, 2)$.

Given $Q \in \mathcal{V} = \{(2^2 \times 2_+^{1+8}).2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.D_8\}$, let $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ be the subfamilies of $\mathcal{R}$ defined by (5.6) with 3 replaced by 2^2, 3^6 by $2^2 \times 2_+^{1+8}$. Then we may suppose (5.7) holds with Ω replaced by $\mathcal{V}$. Thus if $P_1 =_G 2^2$ and $P_2 =_G 2^2 \times 2_+^{1+8}$, then $P_3 \in \Phi((2^2 \times 2_+^{1+8}).U_4(2).2, 2)\backslash\mathcal{V}$; if $P_1 =_G 2^2$, then $P_2 \notin_G \mathcal{V}$.

Case (6.3). By MAGMA, $N_{(2^2\times 2_+^{1+8}).2.2^4.U_4(2).2}(2^{11}.2^4) = N_{M_2}(2^{11}.2^4) = 2^{11}.2^4.S_5$, and we may take

$$\Phi(2^{11}.2^4.S_5, 2) = \{2^{11}.2^4, 2^{11}.2^2.2^3, 2^{11}.2^2.2^4, S\} \subseteq \Phi(M_2, 2)$$

and $N_{2^{11}.2^4.S_5}(R) = N_{M_2}(R)$ for each $R \in \Phi(2^{11}.2^4.S_5, 2)$.

Let $P \in \Phi(2^{11}.2^4.S_5, 2)\backslash\{2^{11}.2^4\}$, and let $\mathcal{H}^+(P)$ and $\mathcal{H}^0(P)$ be the subfamilies of $\mathcal{R}$ defined by (5.12) with 2 replaced by 2^2. Then we may suppose

$$C \notin \bigcup_{P\in\Phi(2^{11}.2^4.S_5,2)\backslash\{2^{11}.2^4\}} (\mathcal{H}^+(P) \cup \mathcal{H}^0(P)).$$

So if $P_1 =_G 2^2$ and $P_2 =_G 2^2 \times 2_+^{1+8}$, then $C \in \{C(19), C(26)\}$.

Case (6.4). As shown in the proof (5) of Lemma 4.4, $N_{M_2}(2^{6+8}) = 2^{6+8}.(3 \times A_5).2.2$ and we may take

$$\Phi^*(N_{M_2}(2^{6+8}), 2) = \{2^{6+8}.2, 2^2.2^{4+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.2^2, 2^{11}.2^2.2^4, 2^{6+8}.D_8\},$$

and $N_{N_{M_2}(2^{6+8})}(R) = N_{M_2}(R)$ for each $R \in \Phi(N_{M_2}(2^{6+8}), 2)$, where $\Phi^*(N_{M_2}(2^{6+8}), 2) = \Phi(N_{M_2}(2^{6+8}), 2)\backslash\{2^{6+8}, S\}$. We may suppose $\Phi(N_{M_2}(2^{6+8}), 2) \subseteq \Phi(M_2, 2)$.

Let $\mathcal{L}^+(2^2.2^{4+8}.2)$ and $\mathcal{L}^0(2^2.2^{4+8}.2)$ be the subfamilies of $\mathcal{R}$ defined by (5.6) with 3 replaced by 2^2, 3^6 by 2^{6+8} and Q by $2^2.2^{4+8}.2$. Then we may suppose

$$C \notin (\mathcal{L}^+(2^2.2^{4+8}.2) \cup \mathcal{L}^0(2^2.2^{4+8}.2)).$$

Thus if $P_1 =_G 2^2$ and $P_2 =_G 2^{6+8}$, then $P_3 \neq_G 2^2.2^{4+8}.2$; if $P_1 =_G 2^2$, then $P_2 \neq_G 2^2.2^{4+8}.2$.

Case (6.5). By MAGMA, $N_{2^{6+8}.(3\times A_5).2.2}(2^{6+8}.2) = N_{M_2}(2^{6+8}.2) = 2^{6+8}.2.S_5$, and we may take
$$\Phi(2^{6+8}.2.S_5, 2) = \{2^{6+8}.2, 2^{6+8}.2^2, 2^{11}.2^2.2^4, S\} \subseteq \Phi(M_2, 2)$$
and $N_{2^{6+8}.2.S_5}(R) = N_{M_2}(R)$ for each $R \in \Phi(2^{6+8}.2.S_5, 2)$.

Let $P \in \Phi(2^{6+8}.2.S_5, 2)\backslash\{2^{6+8}.2\}$, and let $\mathcal{H}^+(P)$ and $\mathcal{H}^0(P)$ be the subfamilies of $\mathcal{R}$ defined by (5.12) with 2 replaced by 2^2, $2^2 \times 2_+^{1+8}$ by 2^{6+8} and $2^{11}.2^4$ by $2^{6+8}.2$. Then we may suppose (5.13) holds. So if $P_1 =_G 2^2$ and $P_2 =_G 2^{6+8}$, then either $C =_G C(28)$ or $P_3 \in_G \{(2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.D_8\}$.

Case (6.6). By MAGMA, $N_{2^{6+8}.(3\times A_5).2.2}((2^2 \times 2_+^{1+8}).2.2^4) = N_{M_2}((2^2 \times 2_+^{1+8}).2.2^4) = (2^2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3)$, and we may take

$$\Phi(N_{M_2}((2^2 \times 2_+^{1+8}).2.2^4), 2) = \{(2^2 \times 2_+^{1+8}).2.2^4, 2^{6+8}.D_8, 2^{11}.2^2.2^4, S\} \subseteq \Phi(M_2, 2)$$

and $N_{N_{M_2}((2^2\times 2_+^{1+8}).2.2^4)}(R) = N_{M_2}(R)$ for each $R \in \Phi(N_{M_2}((2^2 \times 2_+^{1+8}).2.2^4), 2)$.

Let $\mathcal{H}^+(2^{6+8}.D_8)$ and $\mathcal{H}^0(2^{6+8}.D_8)$ be the subfamilies of $\mathcal{R}$ defined by (5.12) with 2 replaced by 2^2, $2^2 \times 2_+^{1+8}$ by 2^{6+8}, $2^{11}.2^4$ by $(2^2 \times 2_+^{1+8}).2.2^4$ and P by $2^{6+8}.D_8$. Then we may suppose (5.13) holds with $P = 2^{6+8}.D_8$.

Let $C' : 1 < 2^2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4 < 2^{11}.2^2.2^4 < S$ and

$$g(C') : 1 < 2^2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4 < S.$$

Then $N(C') = N(g(C')) = N(S)$, so that (5.5) holds and we may suppose $C \neq_G C', g(C')$.

So if $P_1 =_G 2^2$ and $P_2 =_G 2^{6+8}$, then $C \in \{C(21), C(28), C(30), C(31)\}$.

Case (7). Suppose $P_1 = 2 = O_2(M_1)$ and $P_2 = 2^2 = O_2(M_2)$. Then $N_{M_1}(2^2) = 2^2.U_6(2)$ and we may suppose $P_3 \in \Phi(2^2.U_6(2), 2)$, which is given in Case (2.4) above.

Case (7.1). Now $N_{2^2.U_6(2)}(2^2 \times 2_+^{1+8}) = (2^2 \times 2_+^{1+8}).U_4(2)$ and we may take

$$\Phi((2^2 \times 2_+^{1+8}).U_4(2), 2) = \{2^2 \times 2_+^{1+8}, 2^{11}.2^4, (2^2 \times 2_+^{1+8}).2.2^4, S'\},$$

and in addition, $N_{(2^2\times 2_+^{1+8}).U_4(2)}(R) = N_{2^2.U_6(2)}(R)$ for each $R \in \Phi((2^2 \times 2_+^{1+8}).U_4(2), 2)$. We may suppose $\Phi((2^2 \times 2_+^{1+8}).U_4(2), 2) \subseteq \Phi(2^2.U_6(2), 2)$.

Given $P \in \Phi((2^2 \times 2_+^{1+8}).U_4(2), 2)\backslash\{2^2 \times 2_+^{1+8}\}$, let $\mathcal{H}^+(P)$ and $\mathcal{H}^0(P)$ be the subfamilies of $\mathcal{R}$ given by (5.12) with $2^2 \times 2_+^{1+8}$ replaced by 2^2 and $2^{11}.2^4$ by $2^2 \times 2_+^{1+8}$. Then (5.13) holds for each P. So if $P_1 =_G 2$ and $P_2 =_G 2^2$, then $P_3 \notin_G \Phi((2^2 \times 2_+^{1+8}).U_4(2), 2)\backslash\{2^2 \times 2_+^{1+8}\}$ and if moreover, $P_3 =_G 2^2 \times 2_+^{1+8}$, then $C =_G C(20)$.

Case (7.2). By MAGMA, $N_{2^2.U_6(2)}(2^{6+8}) = 2^{6+8}.(3 \times A_5).2$ and we may take

$$\Phi(2^{6+8}.(3 \times A_5).2, 2) = \{2^{6+8}, 2^{6+8}.2, (2^2 \times 2_+^{1+8}).2.2^4, S'\} \subseteq \Phi(2^2.U_6(2), 2)$$

and $N_{2^{6+8}(3\times A_5).2}(R) = N(R)$ for each $S' \neq R \in \Phi(2^{6+8}.(3 \times A_5).2, 2)$. Let $\mathcal{H}^+(2^{6+8}.2)$ and $\mathcal{H}^0(2^{6+8}.2)$ be the subfamilies of $\mathcal{R}$ given by (5.12) with $2^2 \times 2_+^{1+8}$ replaced by 2^2, $2^{11}.2^4$ by 2^{6+8} and P by $2^{6+8}.2$. Then (5.13) holds for $P = 2^{6+8}.2$.

Let $C' : 1 < 2 < 2^2 < 2^{6+8} < S'$ and

$$g(C') : 1 < 2 < 2^2 < 2^{6+8} < (2^2 \times 2_+^{1+8}).2.2^4 < S'.$$

Then $N(C') = N(g(C'))$, so that (5.5) holds and we may suppose $C \neq_G C', g(C')$.

Thus if $P_1 =_G 2$, $P_2 =_G 2^2$ and $P_3 =_G 2^{6+8}$, then $C \in_G \{C(22), C(29)\}$.

Case (7.3). By MAGMA, $N_{2^2.U_6(2)}(2^{11}) = 2^{11}.L_3(4)$ and we may take

$$\Phi(2^{11}.L_3(4), 2) = \{2^{11}, 2^{11}.2^4, 2^{6+8}.2, S'\} \subseteq \Phi(2^2.U_6(2), 2)$$

and $N_{2^{11}.L_3(4)}(R) = N_{2^2.U_6(2)}(R)$ for each $R \in \Phi(2^{11}.L_3(4), 2)$.

Let $C' : 1 < 2 < 2^2 < 2^{11} < S'$ and $g(C') : 1 < 2 < 2^2 < 2^{11} < 2^{6+8}.2 < S'$. Then $N(C') = N(g(C'))$, so that (5.5) holds and we may suppose $C \neq_G C', g(C')$. Thus if $P_1 =_G 2$, $P_2 =_G 2^2$ and $P_3 =_G 2^{11}$, then $C \in_G \{C(18), C(25), C(27), C(32)\}$.

This completes the classification of the radical 2-chains, and the determination of the normalizers of these chains. □

B Proof of Theorem 6.2

PROOF: We may suppose B has a non-cyclic defect group.

(1). Suppose $p = 5$, so that by Lemma 4.5 (a), $B = B_1$ or B_0. Let $C = C(2)$, $C' = C(3)$, so that by Lemma 5.1, $N(C) \simeq F_5^4 \times S_7$ and $N(C') \simeq F_5^4 \times K$, where $K = N_{S_7}(5) = F_5^4 \times 2$. If $b(C) \in \mathrm{Blk}(N(C))$ with $b(C)^G = B$, then $b(C)$ has a defect group 5^2 and $b(C) = b_1 \times b_2$, where $b_1 \in \mathrm{Blk}(F_5^4)$ and $b_2 \in \mathrm{Blk}(S_7)$. In addition, $D(b_1) \simeq D(b_2) \simeq 5$ and $b_1 = B_0(F_5^4)$. Similarly, if $b(C') \in \mathrm{Blk}(N(C'))$ with $b(C')^G = B$, then $b(C') = b_1 \times b_K$ for some $b_K \in \mathrm{Blk}(K)$ with $D(b_K) \simeq 5$. We may suppose $b_K^{S_7} = b_2$. Since b_2 has a cyclic defect group, each character of both b_K and b_2 has height 0 (or defect 2) and $|\mathrm{Irr}(b_K)| = |\mathrm{Irr}(b_2)|$. It follows that

$$\mathrm{k}(N(C), B_j, d) = \mathrm{k}(N(C'), B_j, d)$$

for $j = 0, 1$ and all integers $d \geq 0$. By MAGMA, $N(C(4)) \simeq (F_5^4 \times F_5^4 \times 2).2.3$ has 40 irreducible characters, all of 5-defect 2. In addition, $N(C(4))$ has two blocks, $b_0(C(4)) = B_0(N(C(4)))$ and $b_1(C(4))$, and each has 20 irreducible characters. It follows by Lemma 4.5 (a) that

$$\mathrm{k}(G, B_j, d) = \mathrm{k}(N(C(4))), B_j, d) = \begin{cases} 20 & \text{if } d = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $j = 0, 1$. Thus the result follows when $p = 5$.

(2). Suppose $p = 3$, so that by Lemma 4.5 (b), $B = B_0$. First, we consider the radical 3-chains $C(j)$ with $\mathrm{d}(N(C(j))) = 10$, so that $2 \leq j \leq 9$.

By MAGMA, $N(C(2)) \simeq S_3 \times O_7(3)$ and $N(C(3)) \simeq S_3 \times 3^5{:}U_4(2){:}2$ have 174 and 228 irreducible characters, respectively, whose degrees are given in Tables C–28 and C–29. In addition, $N(C(2))$ has two blocks and the principal block contains 171 irreducible characters.

It follows that

$$\mathrm{k}(N(C),B_0,d)=\begin{cases}54 & \text{if } d=10,\\ 63 & \text{if } d=9,\\ \alpha_1 & \text{if } d=8,\\ 9 & \text{if } d=7,\\ 18 & \text{if } d=6,\\ \alpha_2 & \text{if } d=5,\\ 0 & \text{otherwise,}\end{cases}$$

where $C\in\{C(2),C(3)\}$ and $(\alpha_1,\alpha_2)=(24,3)$ or $(84,0)$ according as $C=C(2)$ or $C(3)$.

The subgroups $N(C(4))\simeq S_3\times 3^{3+3}{:}\,3^2{:}\,2S_4$ and $N(C(5))\simeq S_3\times 3^{3+3}{:}\,L_3(3)$ have 237 and 138 irreducible characters, respectively, whose degrees are given in Tables C–30 and C–31.

It follows that

$$\mathrm{k}(N(C),B_0,d)=\begin{cases}54 & \text{if } d=10,\\ 36 & \text{if } d=9,\\ \beta & \text{if } d=8,\\ 9 & \text{if } d=7,\\ 0 & \text{otherwise,}\end{cases}$$

where $C\in\{C(4),C(5)\}$ and $\beta=138$ or 39 according as $C=C(4)$ or $C(5)$.

The subgroups $N(C(6))\simeq S_3\times 3_+^{1+6}.3{:}\,2S_4$ and $N(C(9))=S''.2^3$ have 219 and 309 irreducible characters, respectively, whose degrees are given in Tables C–32 and C–33.

It follows that

$$\mathrm{k}(N(C),B_0,d)=\begin{cases}54 & \text{if } d=10,\\ 63 & \text{if } d=9,\\ \gamma_1 & \text{if } d=8,\\ 54 & \text{if } d=7,\\ \gamma_2 & \text{if } d=6,\\ 0 & \text{otherwise,}\end{cases}$$

where $C\in\{C(6),C(9)\}$ and $(\gamma_1,\gamma_2)=(39,9)$ or $(138,0)$ according as $C=C(6)$ or $C(9)$.

The subgroups $N(C(8))\simeq S_3\times 3^5{:}\,3^3{:}\,(S_4\times 2)$ and $N(C(7))\simeq S_3\times 3_+^{1+6}.(2A_4\times A_4).2$ have 300 and 252 irreducible characters, respectively, whose degrees are given in Tables C–34 and C–35.

It follows that

$$\mathrm{k}(N(C),B_0,d)=\begin{cases}54 & \text{if } d=10,\\ 90 & \text{if } d=9,\\ \delta_1 & \text{if } d=8,\\ 54 & \text{if } d=7,\\ \delta_2 & \text{if } d=6,\\ \delta_3 & \text{if } d=5,\\ 0 & \text{otherwise,}\end{cases}$$

where $C \in \{C(8), C(7)\}$ and $(\delta_1, \delta_2, \delta_3) = (84, 18, 0)$ or $(24, 27, 3)$ according as $C = C(8)$ or $C(7)$. Thus

$$\sum_{i=2}^{9} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = 0.$$

Next we consider the chain $C(j)$ with $\mathrm{d}(N(C(j))) = 12$, so that $11 \leq j \leq 22$.

The subgroups $N(C(15)) \simeq 3^3.3^6{:}\, L_3(3)$ and $N(C(16)) \simeq 3^3{:}\, 3^6{:}\, 3^2{:}\, 2S_4$ have 139 and 259 irreducible characters, respectively, whose degrees are given in Tables C–36 and C–37.

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 45 & \text{if } d = 12, \\ 15 & \text{if } d = 11, \\ a_1 & \text{if } d = 10, \\ a_2 & \text{if } d = 9, \\ 4 & \text{if } d = 8, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(15), C(16)\}$ and $(a_1, a_2) = (45, 30)$ or $(81, 114)$ according as $C = C(15)$ or $C(16)$.

The subgroups $N(C(20)) \simeq 3_+^{1+8}{:}\, 3^2{:}\, 2S_4$ and $N(C(21)) \simeq S'.2^2$ have 261 and 372 irreducible characters, respectively, whose degrees are given in Tables C–38 and C–39.

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 45 & \text{if } d = 12, \\ 78 & \text{if } d = 11, \\ b_1 & \text{if } d = 10, \\ b_2 & \text{if } d = 9, \\ 54 & \text{if } d = 8, \\ b_3 & \text{if } d = 7, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(20), C(21)\}$ and $(b_1, b_2, b_3) = (45, 30, 9)$ or $(81, 114, 0)$ according as $C = C(20)$ or $C(21)$. Thus

$$\sum_{i \in \{15,16,20,21\}} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = \begin{cases} -9 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases}$$

The subgroups $N(C(11)) \simeq N(C(17)) \simeq 3^3.3^6{:}\, 3^2.(2S_4 \times 2)$ and $N(C(18)) \simeq N(C(22)) \simeq S'.2^3$ have 269 and 372 irreducible characters, respectively, whose degrees are given in Tables C–40 and C–41.

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 54 & \text{if } d = 12, \\ c_1 & \text{if } d = 11, \\ 72 & \text{if } d = 10, \\ 120 & \text{if } d = 9, \\ c_2 & \text{if } d = 8, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(11), C(18)\}$ or $\{C(17), C(22)\}$ and $(c_1, c_2) = (18, 5)$ or $(72, 54)$ according as $C \in \{C(11), C(17)\}$ or $C \in \{C(18), C(22)\}$.

The subgroups $N(C(12)) \simeq 3^3.3^6{:}\,(L_3(3) \times 2)$ and $N(C(13)) = 3_+^{1+8}.3^2{:}\,(2S_4 \times 2)$ have 149 and 261 irreducible characters, respectively, whose degrees are given in Tables C–42 and C–43.

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 54 & \text{if } d = 12, \\ d_1 & \text{if } d = 11, \\ 36 & \text{if } d = 10, \\ 36 & \text{if } d = 9, \\ d_2 & \text{if } d = 8, \\ d_3 & \text{if } d = 7, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(12), C(13)\}$ and $(d_1, d_2, d_3) = (18, 5, 0)$ or $(72, 54, 9)$ according as $C = C(12)$ or $C(13)$.

The subgroups $N(C(14)) \simeq 3^6{:}\,L_4(3){:}\,2$ and $N(C(19)) \simeq 3_+^{1+8}.3.2_+^{1+4}.(S_3 \times S_3)$ have 159 and 271 irreducible characters, respectively, whose degrees are given in Tables C–44 and C–45.

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 54 & \text{if } d = 12, \\ e_1 & \text{if } d = 11, \\ 45 & \text{if } d = 10, \\ 31 & \text{if } d = 9, \\ e_2 & \text{if } d = 8, \\ e_3 & \text{if } d = 7, \\ 6 & \text{if } d = 6, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(14), C(19)\}$ and $(d_1, d_2, d_3) = (18, 5, 0)$ or $(72, 54, 9)$ according as $C = C(14)$ or $C(19)$. Thus

$$\sum_{i=11}^{22} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = \begin{cases} 9 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Finally, we consider the chain $C(j)$ with $\mathrm{d}(N(C(j))) = 13$, so $j \in \{1, 10, 23, 24\}$.

The subgroups $N(C(23)) \simeq 3_+^{1+8}.3_+^{1+2}{:}\,(2S_4 \times 2)$ and $N(C(24)) \simeq 3^3.3.3^3.3^3{:}\,(L_3(3) \times 2)$ have 206 and 128 irreducible characters, respectively, whose degrees are given in Tables C–46 and C–47.

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 27 & \text{if } d = 13, \\ 24 & \text{if } d = 12, \\ f_1 & \text{if } d = 11, \\ 48 & \text{if } d = 10, \\ f_2 & \text{if } d = 9, \\ f_3 & \text{if } d = 8, \\ f_4 & \text{if } d = 7, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(23), C(24)\}$ and $(f_1, f_2, f_3, f_4) = (41, 36, 26, 4)$ or $(8, 18, 2, 1)$ according as $C = C(23)$ or $C(24)$.

Since $N(C(10)) \simeq 3_+^{1+8}{:}\,2_-^{1+6}{:}\,3_+^{1+2}{:}\,2S_4$ is a maximal subgroup of Fi_{23}, its character table and that of $N(C(1)) = \mathrm{Fi}_{23}$ are stored in the GAP library. Thus $N(C(10))$ and the principal block B_0 of Fi_{23} (see Lemma 4.5 (b)) have 181 and 94 irreducible characters, respectively; the degrees of characters of $\mathrm{Irr}(N(C(10)))$ are given in Table C–48.

It follows by Lemma 4.5 (b) that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 27 & \text{if } d = 13, \\ 24 & \text{if } d = 12, \\ g_1 & \text{if } d = 11, \\ 21 & \text{if } d = 10, \\ g_2 & \text{if } d = 9, \\ g_3 & \text{if } d = 8, \\ g_4 & \text{if } d = 7, \\ 6 & \text{if } d = 6, \\ 1 & \text{if } d = 5, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(1), C(10)\}$ and $(g_1, g_2, g_3, g_4) = (8, 4, 2, 1)$ or $(41, 22, 26, 13)$ according as $C = C(1)$ or $C(10)$. Thus

$$\sum_{i \in \{1,10,23,14\}} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = \begin{cases} -9 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases} \qquad (6.2)$$

Theorem 6.2 follows by (6.1) and (6.2). □

C Character tables for chain normalisers

Degree	1	22	56	176	231	252	385	440	560
Number	1	1	3	3	1	1	1	1	1
Degree	616	770	792	1155	1232	1386	1540	2464	3080
Number	4	2	3	3	3	1	1	3	5
Degree	3360	3520	4620	4928	5544	6160	6930	7776	8064
Number	6	1	1	1	4	11	1	3	1
Degree	9240	9856	10395	11264	12320	13608	13860	14784	15840
Number	3	3	5	4	9	3	1	1	3
Degree	18480	18711	20790	21384	22176	24640	25515	27720	30240
Number	3	2	2	3	3	10	2	3	3
Degree	32768	37422	40095	44352					
Number	4	1	1	3					

Table C–1: The degrees of characters in $\mathrm{Irr}(2^2.U_6(2))$

Degree	1	3	4	5	10	15	16	20	30	40
Number	1	2	1	4	2	9	4	1	6	12
Degree	48	60	64	80	96	120	160	192	240	320
Number	8	2	4	31	6	24	8	6	8	4

Table C–2: The degrees of characters in $\mathrm{Irr}(2^{11}.2^4.A_5)$

Degree	1	3	4	5	15	16	20	30	40
Number	2	4	2	2	8	6	6	8	12
Degree	48	60	64	80	120	160	240	320	
Number	12	16	6	30	16	4	19	2	

Table C–3: The degrees of characters in $\mathrm{Irr}(2^{6+8}.2.A_5)$

Degree	1	2	3	4	6	8	9	12	16	18
Number	6	3	2	6	4	3	8	2	60	1
Degree	24	32	36	48	64	72	96	128	144	192
Number	25	30	16	21	24	12	22	12	20	2

Table C–4: The degrees of characters in $\mathrm{Irr}((2^2 \times 2_+^{1+8}).2.2^4.3.S_3)$

Degree	1	20	21	35	45	56	63	64	84	105
Number	1	1	1	3	2	3	4	1	1	1
Degree	120	210	280	315	336	360	420	448	504	560
Number	3	2	10	4	3	6	1	6	3	4
Degree	630	720	840	960	1008	1260	1344	1680	2240	
Number	6	3	3	3	6	1	3	3	1	

Table C–5: The degrees of characters in $\mathrm{Irr}(2^{11}.L_3(4))$

Degree	1	5	6	10	15	16	20	24	30	40
Number	1	2	1	2	2	4	1	1	3	8
Degree	45	60	64	80	81	96	120	135	160	216
Number	2	1	1	11	1	4	10	2	20	6
Degree	240	270	320	360	384	405	480	540	640	720
Number	21	1	10	8	4	6	18	2	8	10
Degree	810	864	960	1024	1080	1296				
Number	1	6	5	4	7	7				

Table C–6: The degrees of characters in $\mathrm{Irr}((2^2 \times 2_+^{1+8}).U_4(2))$

Degree	1	2	3	4	5	6	8	10	16	20	30
Number	2	1	4	2	2	2	1	1	6	6	3
Degree	32	40	45	48	60	64	80	90	96	120	128
Number	3	15	8	12	8	6	36	1	6	4	3
Degree	160	180	240	320	360	480	640				
Number	19	8	15	4	12	6	1				

Table C–7: The degrees of characters in $\mathrm{Irr}(2^{6+8}(3 \times A_5).2)$

Degree	1	3	4	6	12	16	24	48	64	96
Number	6	10	6	8	18	60	36	39	24	18

Table C–8: The degrees of characters in $\mathrm{Irr}(S'.3)$

Degree	1	21	45	55	77	99	154	176	210
Number	1	1	2	1	1	1	1	2	1
Degree	231	280	330	385	616	672	693	770	990
Number	1	2	1	3	4	1	1	1	2
Degree	1056	1155	1760	1980	2310	2464	2640	3360	3465
Number	2	2	4	1	4	4	3	2	2
Degree	3696	4620	5544	6160	6720	6930	7392	8064	9856
Number	2	1	2	5	2	2	1	2	1

Table C–9: The degrees of characters in Irr($2^{11}.M_{22}$)

Degree	1	6	10	15	16	20	24	30	32	40
Number	2	2	1	4	4	3	2	2	1	4
Degree	60	64	80	81	90	96	120	135	160	192
Number	3	2	5	2	1	4	8	4	5	1
Degree	216	240	270	320	384	405	432	480	540	640
Number	4	17	2	14	4	4	2	15	4	4
Degree	720	768	810	864	960	1024	1080	1280	1296	1440
Number	4	1	4	4	13	4	6	4	6	5
Degree	1728	1920	2048	2160	2592					
Number	2	1	1	2	2					

Table C–10: The degrees of characters in Irr($(2^2 \times 2_+^{1+8}).U_4(2).2$)

Degree	1	2	5	8	9	10	16	18	20	32
Number	2	1	4	4	2	4	4	1	1	1
Degree	45	60	80	90	96	120	128	135	144	160
Number	4	4	8	4	4	6	4	4	2	14
Degree	180	192	240	256	270	288	320	360	384	480
Number	5	2	9	2	2	9	5	2	4	10
Degree	540	576	720	768	960	1080	1440			
Number	8	4	10	2	7	2	5			

Table C–11: The degrees of characters in Irr($2^{6+8}(S_3 \times A_6)$)

Degree	1	2	3	4	6	8	12	16	24	32	48	64	96	128	192
Number	4	2	12	4	10	2	22	16	32	26	37	8	33	10	4

Table C–12: The degrees of characters in Irr($2^{11}.2^2.2^4.S_3$)

Degree	1	78	352	429	1001	1430	2080
Number	1	1	1	1	1	1	2
Degree	3003	3080	5824	10725	13650	13728	27456
Number	1	1	4	1	1	2	1
Degree	30030	32032	43680	45045	48048	50050	75075
Number	1	1	1	1	3	2	3
Degree	81081	105600	114400	123200	133056	138600	146432
Number	1	2	1	1	1	2	2
Degree	150150	205920	228800	235872	289575	300300	320320
Number	1	1	1	2	1	1	3
Degree	360855	370656	400400	436800	450450	480480	576576
Number	1	1	6	2	2	2	1
Degree	577368	579150	582400	600600	675675	686400	720720
Number	1	1	2	3	1	2	1
Degree	800800	852930	915200	938223	972972	982800	1029600
Number	1	1	2	1	1	2	1
Degree	1201200	1297296	1360800	1372800	1441792	1663200	1791153
Number	3	2	1	1	2	2	1
Degree	1876446	2027025	2050048	2196480	2316600	2358720	2402400
Number	1	1	1	2	1	2	3
Degree	2555904	2729376					
Number	4	1					

Table C–13: The degrees of characters in $\mathrm{Irr}(2.\mathrm{Fi}_{22})$

Degree	1	5	8	9	10	15	16	30	45
Number	2	4	4	2	2	4	2	4	4
Degree	60	80	90	96	120	128	144	160	180
Number	5	8	6	4	4	4	2	10	13
Degree	240	288	360	384	480	720	960		
Number	9	8	6	4	12	7	2		

Table C–14: The degrees of characters in $\mathrm{Irr}(2^{6+8}.2.A_6)$

Degree	1	2	3	4	6	8	9	12	16	18	24
Number	4	4	4	5	4	4	8	5	17	4	14
Degree	32	36	48	64	72	96	128	144	192	256	288
Number	34	16	23	21	12	23	14	16	11	5	7

Table C–15: The degrees of characters in $\mathrm{Irr}((2^2 \times 2_+^{1+8})2.2^4.(S_3 \times S_3))$

Degree	1	4	5	6	10	15	16	20	30	32	40	60
Number	2	2	4	1	5	10	4	2	6	1	8	6
Degree	64	80	96	120	128	160	192	240	320	384	480	640
Number	4	18	8	20	1	20	2	15	6	3	2	1

Table C–16: The degrees of characters in $\mathrm{Irr}(2^{11}.2^4.S_5)$

Degree	1	20	21	35	45	56	64	70	84	105
Number	2	2	2	2	4	2	2	1	2	2
Degree	112	120	126	210	240	280	315	336	360	420
Number	1	2	2	4	1	8	4	2	4	2
Degree	504	560	630	672	720	840	896	960	1008	1120
Number	2	7	5	1	4	2	3	2	1	1
Degree	1260	1344	1440	1680	1920	2016	2240	2688	3360	
Number	4	2	1	3	1	3	2	1	1	

Table C–17: The degrees of characters in $\mathrm{Irr}(2^{11}.L_3(4).2)$

Degree	1	2	4	5	6	8	10	12	16	20	30
Number	4	2	4	4	2	2	2	1	4	4	2
Degree	32	40	45	60	64	80	90	96	120	128	160
Number	4	12	8	9	5	21	4	6	6	4	28
Degree	180	192	240	256	320	360	480	640	720	960	
Number	8	3	7	1	14	10	10	2	4	2	

Table C–18: The degrees of characters in $\mathrm{Irr}(2^{6+8}(3 \times A_5).2.2)$

Degree	1	22	56	112	176	231	252	352	385
Number	2	2	2	1	2	2	2	1	2
Degree	440	560	616	792	1155	1232	1386	1540	1584
Number	2	2	4	2	2	3	2	3	1
Degree	2310	2464	3080	3520	4620	4928	5544	6160	6720
Number	1	3	2	2	2	3	4	4	3
Degree	6930	7776	8064	9240	9856	10395	11088	11264	12320
Number	2	2	2	2	2	2	1	4	7
Degree	13608	13860	14784	15552	15840	18480	18711	19712	20790
Number	2	2	2	1	2	3	4	1	2
Degree	21384	22176	22528	24640	27216	27720	30240	31680	32768
Number	2	2	1	8	1	2	2	1	4
Degree	36960	37422	40095	41580	42768	44352	49280	51030	55440
Number	1	2	2	1	1	3	4	1	1
Degree	60480	65536	88704						
Number	1	1	1						

Table C–19: The degrees of characters in $\mathrm{Irr}(2^2.U_6(2).2)$

Degree	1	4	5	6	15	16	20	30	32	40
Number	4	4	4	2	8	4	4	10	2	10
Degree	60	64	80	96	120	128	160	240	320	480
Number	18	4	16	6	16	2	20	15	4	7

Table C–20: The degrees of characters in $\mathrm{Irr}(2^{6+8}.2.S_5)$

Degree	1	6	10	14	15	21	30	35	45
Number	2	2	4	4	4	2	1	2	4
Degree	90	105	112	120	140	180	210	240	280
Number	4	2	2	2	4	1	3	1	8
Degree	336	420	560	630	840	896	1008	1120	1260
Number	4	9	4	2	6	4	2	2	4
Degree	1344	1680	2016	2520	3360				
Number	4	7	2	2	1				

Table C–21: The degrees of characters in $\mathrm{Irr}(2^{6+8}.2.A_7)$

Degree	1	2	3	4	6	8	9	12	16	18
Number	4	10	4	8	6	10	4	9	8	4
Degree	24	27	32	36	48	64	72	96	108	128
Number	6	4	10	9	19	8	14	26	8	10
Degree	144	192	216	256	288	384	432	576	768	864
Number	17	19	8	4	8	6	8	1	1	1

Table C–22: The degrees of characters in $\mathrm{Irr}((2^2 \times 2_+^{1+8})2.2^4.3.(S_3 \times S_3))$

Degree	1	2	4	5	6	8	10	15	16	20	30
Number	2	1	2	4	3	1	9	6	2	4	3
Degree	32	40	45	48	60	64	80	90	96	120	128
Number	1	1	4	2	4	2	4	4	3	16	1
Degree	160	192	240	288	320	360	480	640	720	960	1152
Number	9	2	17	5	4	14	8	1	4	2	1

Table C–23: The degrees of characters in $\mathrm{Irr}(2^{11}.2^4.3.S_5)$

Degree	1	22	45	230	231	253	506	770	896
Number	1	1	2	1	3	2	1	2	2
Degree	990	1035	1288	1518	2024	2530	3542	3795	5313
Number	2	1	1	1	1	2	3	2	1
Degree	7084	8855	10120	10626	11385	12880	14168	17710	20608
Number	1	1	1	3	2	3	2	1	2
Degree	22770	26565	28336	30360	32384	35420	56672	57960	70840
Number	3	1	1	1	1	1	1	1	1

Table C–24: The degrees of characters in $\mathrm{Irr}(2^{11}.M_{23})$

Degree	1	2	6	10	12	15	16	20	24	30
Number	2	1	2	3	1	4	2	5	2	4
Degree	32	40	48	60	64	80	81	90	96	120
Number	1	1	3	6	2	3	2	3	2	7
Degree	128	135	160	162	192	240	270	288	320	360
Number	1	4	3	1	1	9	4	2	5	6
Degree	384	480	540	640	648	720	768	810	960	1024
Number	2	6	5	1	4	11	1	2	13	2
Degree	1080	1152	1215	1280	1296	1440	1620	1920	2048	2160
Number	4	2	4	3	2	13	1	3	1	2
Degree	2592	2880	3072	3240	3840	3888	4320			
Number	5	3	2	4	1	4	1			

Table C–25: The degrees of characters in $\mathrm{Irr}((2^2 \times 2_+^{1+8}).(3 \times U_4(2)).2)$

Degree	1	2	6	10	12	14	15	20	21	28
Number	2	1	2	4	1	4	2	2	2	2
Degree	30	35	42	45	70	112	135	140	210	224
Number	1	2	1	2	1	2	4	4	2	1
Degree	270	280	315	336	360	420	560	630	672	840
Number	2	10	2	4	2	8	8	2	2	4
Degree	896	1008	1120	1260	1344	1680	1792	2016	2240	2520
Number	4	2	4	5	4	7	2	3	1	2
Degree	2688	3360	3780	4032						
Number	2	6	4	1						

Table C–26: The degrees of characters in $\mathrm{Irr}(2^{6+8}.(S_3 \times A_7))$

Degree	1	2	3	4	6	8	9	12	16	18	24
Number	4	8	8	4	8	8	4	14	4	4	12
Degree	32	36	48	64	72	96	128	144	192	288	384
Number	8	8	21	4	22	21	8	20	10	9	2

Table C–27: The degrees of characters in $\mathrm{Irr}(2^{11}.2^2.2^4.F_{3^2}^2)$

Degree	1	2	78	91	105	156	168	182	195
Number	2	1	2	2	2	1	2	3	2
Degree	210	260	273	336	364	390	520	546	819
Number	1	4	2	1	1	1	2	3	2
Degree	910	1092	1365	1560	1638	1820	2106	2184	2457
Number	4	3	4	4	3	4	2	3	2
Degree	2730	2835	3120	3276	3640	4095	4212	4368	4536
Number	6	2	2	1	1	4	1	3	2
Degree	4914	5265	5460	5670	5824	6552	7020	7280	7371
Number	1	2	8	1	4	2	4	6	2
Degree	8190	8736	9072	10530	10920	11648	13104	14040	14560
Number	6	1	1	1	3	4	1	2	3
Degree	14742	16380	16640	17472	17920	19683	21840	22113	23296
Number	3	4	4	4	4	2	2	2	1
Degree	29484	32760	33280	34944	35840	39366	43680	44226	
Number	1	1	2	2	2	1	1	1	

Table C–28: The degrees of characters in $\mathrm{Irr}(S_3 \times O_7(3))$

Degree	1	2	6	10	12	15	20	24	30	40
Number	4	2	4	2	2	8	7	4	8	3
Degree	48	60	64	72	80	81	90	120	128	144
Number	2	8	4	4	14	4	14	3	2	2
Degree	160	162	180	240	360	480	540	640	648	720
Number	7	2	13	4	23	10	6	6	4	20
Degree	810	960	1080	1152	1280	1296	1440	1620	1920	2304
Number	4	8	3	2	3	2	5	2	2	1

Table C–29: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^5{:}U_4(2){:}2)$

Degree	1	2	3	4	6	8	12	16	18	24
Number	4	8	4	5	8	17	9	16	20	3
Degree	32	36	48	54	72	96	108	144	288	
Number	4	28	8	6	45	4	3	36	9	

Table C–30: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^{3+3}{:}3^2{:}2S_4)$

Degree	1	2	12	13	16	24	26	27	32
Number	2	1	2	2	8	1	13	2	4
Degree	39	52	54	78	104	156	208	234	312
Number	2	12	1	9	3	10	6	14	3
Degree	416	468	624	702	936	1248	1404		
Number	3	19	6	4	6	3	2		

Table C–31: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^{3+3}{:}\, L_3(3))$

Degree	1	2	3	4	6	8	12	16	24	32
Number	4	14	4	14	4	8	1	10	24	4
Degree	48	54	72	96	108	144	162	216	288	324
Number	24	18	24	6	27	14	6	9	1	3

Table C–32: The degrees of characters in $\mathrm{Irr}(S_3 \times 3_+^{1+6}.3{:}\, 2S_4)$

Degree	1	2	4	6	8	12	18	24	36	54	72	108
Number	8	24	18	28	4	28	72	7	56	36	10	18

Table C–33: The degrees of characters in $\mathrm{Irr}(S''.2^3)$

Degree	1	2	3	4	6	8	12	16	18	24	32
Number	8	20	8	8	16	4	24	10	8	25	4
Degree	36	48	54	72	96	108	144	162	288	324	
Number	30	14	36	33	3	18	12	12	1	6	

Table C–34: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^5{:}\, 3^3{:}\, (S_4 \times 2))$

Degree	1	2	3	4	6	8	9	12	18	24
Number	4	14	8	14	12	4	4	6	2	9
Degree	32	48	54	64	96	108	128	144	162	192
Number	4	22	18	10	21	27	4	8	12	10
Degree	216	288	324	384	486	576	648	972		
Number	9	8	12	2	2	2	3	1		

Table C–35: The degrees of characters in $\mathrm{Irr}(S_3 \times 3_+^{1+6}.(2A_4 \times A_4).2)$

Degree	1	12	13	16	26	27	39	52	78
Number	1	1	1	4	15	1	1	12	4
Degree	208	234	468	624	702	1404	1872	2106	
Number	12	15	15	9	17	12	15	4	

Table C–36: The degrees of characters in Irr($3^3.3^6\colon L_3(3)$)

Degree	1	2	3	4	8	16	18	36
Number	2	3	2	1	26	13	27	27
Degree	48	54	108	144	162	216	432	
Number	13	21	12	27	4	54	27	

Table C–37: The degrees of characters in Irr($3^3\colon 3^6\colon 3^2\colon 2S_4$)

Degree	1	2	3	4	6	8	16	24
Number	2	15	2	13	4	2	13	24
Degree	48	72	144	162	216	324	432	486
Number	48	30	15	27	24	27	6	9

Table C–38: The degrees of characters in Irr($3_+^{1+8}\colon 3^2\colon 2S_4$)

Degree	1	2	4	6	12	18	36	54	108	162
Number	4	28	13	26	52	54	27	78	36	54

Table C–39: The degrees of characters in Irr($S'.2^2$)

Degree	1	2	3	4	8	16	18	32	36	48	54
Number	4	6	4	2	20	18	14	4	24	10	18
Degree	72	96	108	144	162	216	288	324	432	864	
Number	10	4	18	14	4	39	10	1	36	9	

Table C–40: The degrees of characters in Irr($3^3.3^6\colon 3^2\colon (2S_4 \times 2)$)

Degree	1	2	4	6	8	12	18	24	36	54	72	108	162	216	324
Number	8	24	18	20	4	32	28	20	34	60	10	48	36	12	18

Table C–41: The degrees of characters in Irr($S'.2^3$)

Degree	1	12	13	16	26	27	39	52	78
Number	2	2	2	8	14	2	2	12	4
Degree	104	156	208	234	416	468	624	702	936
Number	4	1	8	6	4	12	6	14	6
Degree	1248	1404	1872	2106	2808	3744	4212		
Number	3	17	6	4	3	6	1		

Table C–42: The degrees of characters in $\mathrm{Irr}(3^3.3^6{:}\,(L_3(3)\times 2))$

Degree	1	2	3	4	6	8	12	16	24	32	48	72
Number	4	14	4	14	4	8	1	10	16	4	28	12
Degree	96	144	162	216	288	324	432	486	648	864	972	
Number	19	18	18	24	6	27	10	6	9	2	3	

Table C–43: The degrees of characters in $\mathrm{Irr}(3_+^{1+8}.3^2{:}\,(2S_4\times 2))$

Degree	1	26	39	52	65	90	234	260
Number	2	4	2	2	4	2	8	12
Degree	351	390	416	468	520	585	729	780
Number	2	2	4	2	3	4	2	2
Degree	832	1040	1170	1280	1404	1560	2080	2340
Number	1	8	4	2	4	3	3	7
Degree	3510	4680	5616	6240	7020	8320	9360	10530
Number	8	5	4	8	8	6	4	4
Degree	12480	14040	14976	16640	18720	18954	21060	
Number	1	5	4	3	5	4	1	

Table C–44: The degrees of characters in $\mathrm{Irr}(3^6{:}\,L_4(3){:}\,2)$

Degree	1	2	4	6	8	9	12	16	18	24	32
Number	4	10	8	2	10	4	4	4	2	8	4
Degree	48	64	72	96	128	144	162	192	216	288	324
Number	12	10	16	13	4	8	18	18	8	8	9
Degree	384	432	576	648	864	972	1152	1296	1458	1728	
Number	15	12	6	18	9	9	1	9	6	2	

Table C–45: The degrees of characters in $\mathrm{Irr}(3_+^{1+8}.3.2_+^{1+4}.(S_3\times S_3))$

Degree	1	2	3	4	6	8	12	16	18
Number	4	6	4	2	6	8	6	6	4

Degree	32	36	48	54	72	108	144	162	216
Number	1	6	8	4	16	4	12	8	25

Degree	288	324	432	486	648	864	972	1296	1458
Number	3	6	14	14	12	1	12	10	4

Table C–46: The degrees of characters in $\mathrm{Irr}(3_+^{1+8}.3_+^{1+2}{:}\,(2S_4 \times 2))$

Degree	1	12	13	16	26	27	39	52	54
Number	2	2	2	8	8	2	2	3	1

Degree	78	104	156	208	234	416	468	624	648
Number	8	1	6	2	2	1	3	6	1

Degree	702	864	936	1404	1458	2106	4212	5616	6318
Number	13	4	3	16	1	11	6	12	2

Table C–47: The degrees of characters in $\mathrm{Irr}(3^3.3.3^3.3^3{:}\,(L_3(3) \times 2))$

Degree	1	2	3	4	6	8	12	16	18	24
Number	2	3	2	1	3	4	3	4	1	2

Degree	27	32	36	48	54	64	72	81	96	108
Number	2	1	4	3	3	2	4	2	3	1

Degree	128	144	162	288	324	486	512	576	864	972
Number	1	2	3	2	3	7	4	6	4	6

Degree	1024	1152	1296	1458	1728	2048	2304	2592	3072	3456
Number	4	8	6	2	4	1	4	5	8	5

Degree	3888	4374	4608	5832	6912	7776	8748	9216	10368	11664
Number	7	3	5	6	2	6	3	4	3	5

Degree	13122	18432
Number	1	1

Table C–48: The degrees of characters in $\mathrm{Irr}(3_+^{1+8}{:}\,2_-^{1+6}{:}\,3_+^{1+2}{:}\,2S_4)$