On p-groups having the minimal number of conjugacy classes of maximal size

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Abstract

A long-standing question is the following: do there exist p-groups of odd order having precisely $p-1$ conjugacy classes of the largest possible size? We exhibit a 3-group with this property.

The number of non-identity conjugacy classes of a given size in a p-group, for p a prime, is divisible by $p-1$ (see, for example, [4]).

In 1982 I.D. Macdonald [3] considered finite 2-groups having a unique conjugacy class of maximal size. More generally he suggested the following problem.

> Is there a prime p and a finite p-group G which has precisely $p-1$ classes of maximal size?

As usual, we say that a finite p-group has breadth b when its conjugacy classes of maximal size have p^b elements. Recall that the breadth of a group of order p^n, for $n > 1$, is at most $n-2$ and that the breadth is $n-2$ for, and only for, groups of maximal nilpotency class. A p-group G of maximal nilpotency class and order at least p^4 has $(p-1)^2$ or p^2-p conjugacy classes of maximal size depending on whether G is exceptional or not (see [2, 14.13]). In particular, all 3-groups with maximal nilpotency class and order at least 3^4 have 6 conjugacy classes of maximal size ([2, 14.17]).

Let G be a group of order p^n and breadth b having precisely $p-1$ conjugacy classes of maximal size. Macdonald [3] proved that the nilpotency class of G is at least 3, $b \geq 4$ and $n \leq b^2+b$. He also exhibited a group of order 2^7 having nilpotency class 3 and breadth 4 which has a unique conjugacy class of size 2^4. This is a smallest example, as all groups with order dividing 2^6 have at least two conjugacy classes of maximal size. There are 46 groups with order 2^7 which have

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a unique conjugacy class of maximal size; these groups all have breadth 4. In 2004 Avinoam Mann [4] showed that $n \leq b^2-1$, and proved that $b \geq 5$ for primes $p \geq 5$.

We exhibit the first example of a p-group of odd order having precisely $p-1$ conjugacy classes of maximal size (Corollary 3). It is a 5-generator group of order 3^{15}, having nilpotency class 4, breadth 10 and precisely two conjugacy classes of size 3^{10}.

The example is constructed in two stages. The first stage applies to all odd primes and the second to all primes. Unfortunately 3 is the only prime for which the two can be combined.

We say that the conjugacy class of an element x in a finite p-group G is *special* if it coincides with $x\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G. A special conjugacy class is always of maximal size. Moreover, if G has a special conjugacy class, then all the conjugacy classes of maximal size are special.

Theorem 1. *Let p denote an odd prime. There is a group of order p^{2p+2} with exactly $(p-1)^2$ special conjugacy classes.*

Proof. Let $V = t\,\mathbb{F}_p[t]/(t^{p+2})$; viewed as an algebra, vector space or abelian group as appropriate. Thus V is a $(p+1)$-dimensional vector space over $\mathbb{F}_p$. For each $f \in \mathbb{F}_p[t]$, denote by ϕ_f the $\mathbb{F}_p$-linear map on V which maps v to fv. Let $M = \{\phi_{1+f} \mid f \in t\,\mathbb{F}_p[t]\}$. It is easy to see that M is a subgroup of $\mathrm{End}_{\mathbb{F}_p}(V)$ and that M is an abelian group of order p^p and rank $p-1$. Let α be the algebra isomorphism of V which maps t to $t/(1-t)$. Then α has order p. Let H be the subgroup of $\mathrm{End}_{\mathbb{F}_p}(V)$ generated by α and M. The order of H is p^{p+1} and $M \lhd H$.

Let G be the split extension of V by H. The order of G is p^{2p+2}. We write the elements of G in the form (η, v) or often simply ηv. Observe that G is generated by three elements: α, ϕ_{1+t} and $\bar{t} = t + (t^{p+2}) \in V$. We first prove that $(\alpha^i, k\bar{t})\Phi(G)$ is a special conjugacy class for $1 \leq i, k \leq p-1$. It is enough to show that the centraliser of $\alpha^i\bar{t}$ has order p^3. Note that $C_{G/V}(\alpha^i\bar{t}) = \langle \alpha, \phi_{1+t^p}\rangle V$ and

$$[\langle \alpha, \phi_{1+t^p}\rangle V, \alpha^i\bar{t}\,] = [\langle \phi_{1+t^p}\rangle V, \alpha^i\bar{t}\,] = \langle t^p\bar{t}\,\rangle + [V, \alpha] = tV.$$

Hence

$$|C_G(\alpha^i\bar{t})| = |C_{C_{G/V}(\alpha^i\bar{t})}(\alpha^i\bar{t})| = \frac{|C_{G/V}(\alpha^i\bar{t}\,)|}{|[C_{G/V}(\alpha^i\bar{t}\,), \alpha^i\bar{t}\,]|} = p^3.$$

Second we show G has no other special conjugacy classes. Let $g = (\alpha^i(\phi_{1+t})^j, k\bar{t}\,)$ with $0 \leq i, j, k \leq p-1$ be in a special conjugacy class. The image of a special conjugacy class of G in the quotient H is also special. Hence, since p is odd, we deduce that $i \neq 0$. If $j \neq 0$, then there exists $s \in \{1, \ldots, p-1\}$ such that $si + j \equiv 0 \pmod{p}$. Then

$$C_{G/t^{s+1}V}(g) = \langle g, \phi_{1+t^p}, t^{s-1}\bar{t}, t^s\bar{t}\,\rangle(t^{s+1}V)$$

has order p^4. Consider finally the case $j = 0$. If $k = 0$, then

$$C_G(g) = \langle g, \phi_{1+t^p}, t^{p-1}\bar{t}, t^p\bar{t} \rangle$$

has order p^4. Thus $k \neq 0$ and G has $(p-1)^2$ special conjugacy classes. $\square$

The corresponding 2-group has 3 special conjugacy classes.

Note that all groups with order dividing 3^7, and at least 3^2, have at least 6 conjugacy classes of maximal size.

Theorem 2. *If there is a finite non-cyclic p-group with exactly $2(p-1)$ special conjugacy classes, then there is a finite p-group with exactly $p-1$ special conjugacy classes.*

Proof. Let G be a finite non-cyclic p-group with exactly $2(p-1)$ special conjugacy classes. The special conjugacy classes of G can be taken as $a^i\Phi(G), u^i\Phi(G)$ for $1 \leq i \leq p-1$. Let $p^m = |C_G(a)\Phi(G) : \Phi(G)|$. If $m > 1$, then there is a subgroup M of index p^{m-1} in G containing $\langle a, \Phi(G)\rangle$ such that $G = MC_G(a)$. Note that $a^M = a^G$. So a^M is a special conjugacy class in M and $\Phi(M) = \Phi(G)$. Hence $|C_M(a)\Phi(M) : \Phi(M)| = p$ and the number of special conjugacy classes in M is $2(p-1)$ or $p-1$. Thus it remains to consider the case $m = 1$. Let N be a subgroup of index p^2 in G containing $\Phi(G)$ such that $G = \langle N, a, u\rangle$. Let K be the subgroup of the direct product $G \times G$ generated by $N \times N$, $(a, 1)$, $(1, a)$ and (u, u). We show that K has exactly $p-1$ special conjugacy classes. Note that $\Phi(K) = \Phi(G \times G)$. Thus a special conjugacy class of K is also a special conjugacy class of $G \times G$. The special conjugacy classes of $G \times G$ which lie in K are $(a^i, a^j)\Phi(G \times G)$ and $(u, u)^i\Phi(G \times G)$ for $1 \leq i, j \leq p-1$. However, $C_K((a^i, a^j)) = C_G(a) \times C_G(a)$ has order greater than $|K : \Phi(K)|$ and so the $(a^i, a^j)\Phi(K)$ are not special conjugacy classes of K. On the other hand, $|C_K((u, u))| = |C_G(u)|^2/p = |K : \Phi(K)|$ and so $(u, u)^i\Phi(K)$ are special conjugacy classes of K. $\square$

Corollary 3. *There is a group of order 3^{15} with just two special conjugacy classes.*

The direct product of two copies of the group of order 3^8 in Theorem 1 has 72 maximal subgroups with two special conjugacy classes. We record a power-commutator presentation (see [5, Chapter 9]) for one of these – where all powers and commutators not listed are to be taken as trivial. It can readily be verified using, for example, Magma [1] that the group has exactly two conjugacy classes of size 3^{10} and these are special.

$$
\begin{aligned}
\{x_1, \ldots, x_{15} \quad | \\
& x_2^3 = x_{11}, x_3^3 = x_{12}^2, x_5^3 = x_{13}^2 x_{15}^2, \\
& [x_2, x_1] = x_6, [x_3, x_1] = x_{11}x_{14}, [x_3, x_2] = x_7, [x_4, x_3] = x_8, \\
& [x_5, x_1] = x_{10}^2 x_{11}^2, [x_5, x_2] = x_{10}x_{11}x_{14}, [x_5, x_3] = x_8^2 x_9^2 x_{15}, [x_5, x_4] = x_8 x_{13}^2 x_{15}, \\
& [x_6, x_1] = x_{10}^2, [x_6, x_2] = x_{10}x_{11}^2, [x_6, x_3] = x_{11}x_{14}, [x_6, x_5] = x_{14}^2, \\
& [x_7, x_1] = x_{11}^2, [x_7, x_2] = x_{11}^2, [x_7, x_5] = x_{14}, [x_7, x_6] = x_{14}, \\
& [x_8, x_4] = x_{12}^2, [x_8, x_5] = x_{12}x_{13}, \\
& [x_9, x_1] = x_{14}, [x_9, x_3] = x_{13}^2 x_{15}^2, [x_9, x_4] = x_{13}^2 x_{15}, [x_9, x_5] = x_{13}^2, [x_9, x_8] = x_{15}, \\
& [x_{10}, x_2] = x_{14}^2, [x_{10}, x_3] = x_{14}, [x_{11}, x_1] = x_{14}^2, \\
& [x_{12}, x_5] = x_{15}, [x_{13}, x_3] = x_{15}, [x_{13}, x_4] = x_{15}^2\} \ .
\end{aligned}
$$

There are other examples of order 3^{15}. Given a 3-group K with two special conjugacy classes, Theorem 2 can be used to construct another from the direct product $K \times K$ which has four special conjugacy classes. Hence one gets infinitely many examples.

This construction does not provide examples of p-groups having $p-1$ special conjugacy classes for $p > 3$. Indeed, for $p > 3$ we have not been able to construct a p-group with fewer than $(p-1)^2$ conjugacy classes of maximal size.

We add a remark about 2-groups. A family of 2-groups with just one maximal conjugacy class and with unbounded nilpotency class, like the examples of Fernández-Alcober and O'Brien mentioned in [4], can be constructed in an analogous way to the method of Theorem 2. Let D be the direct product of the unique pro-2 group of maximal class and a maximal class group of order 2^4. There is a maximal subgroup of D which has coclass 4 and has for each $n \geq 7$ a finite 2-quotient of order 2^n with just one special conjugacy class.

Our search for examples was substantially assisted by computer calculations using MAGMA.

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