The class-breadth conjecture revisited

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Dedicated to Charles Leedham-Green on the occasion of his 65th birthday

Abstract

The class-breadth conjecture for groups with prime-power order was formulated by Leedham-Green, Neumann and Wiegold in 1969. We construct a new counter-example to the conjecture: it has order 2^{19} and is a quotient of a 4-dimensional 2-uniserial space group. We translate the conjecture to p-uniserial space groups, prove that these have finite cobreadth, and provide an explicit upper bound. We develop an algorithm to decide the conjecture for p-uniserial space groups, and use this to show that all 3-uniserial space groups of dimension at most 54 satisfy the conjecture. We show that over every finite field there are Lie algebras which fail the corresponding conjecture.

1 Introduction

Leedham-Green, Neumann and Wiegold formulated the class-breadth conjecture in 1969 [12] as part of a study of the relationship between the breadth and the nilpotency class of p-groups. Recall that the breadth $b(G)$ of a p-group G describes the size $p^{b(G)}$ of the largest conjugacy class of G. They conjectured that, for a p-group G, the nilpotency class $c(G)$ is at most $b(G) + 1$. They proved that $c(G) \leq \frac{p}{p-1}b(G) + 1$; more recently, Cartwright [3] proved that $c \leq \frac{5}{3}b(G) + 1$.

The conjecture has been established under various conditions. For example it holds for groups with maximal class, when $b(G) \leq 4$, when $b(G) \leq p + 1$, for metabelian groups, and for groups not covered by certain 2-step centralisers; see [7] for appropriate references. Further it holds for the groups with order dividing 512.

In the 1970s Leedham-Green pioneered the use of coclass as a primary invariant in the theory of p-groups. Recall that the coclass $cc(G)$ of a group G with order p^n is $n - c(G)$. An account of the spectacular progress in this direction is given in the recent book by Leedham-Green and McKay [11]. Similarly, we define the cobreadth

Eick was partially supported by a Feodor Lynen Fellowship from the Alexander von Humboldt Foundation. All three authors were partially supported by the Marsden Fund of New Zealand via grant UOA412. We thank the referee for helpful comments and suggestions.

$cb(G)$ of a group G with order p^n as $n - b(G)$. The class-breadth conjecture can now be formulated as $cb(G) \leq cc(G) + 1$.

An early success for the coclass approach was its use in the construction of counter-examples to the conjecture. Felsch, Neubüser and Plesken [7] prove that, for each positive integer k, there is a 2-group G with $cc(G) + k < cb(G)$. These groups were constructed as quotients of 2-uniserial space groups (see Section 2 for a definition). The smallest has order 2^{34}, coclass 5 and cobreadth 7, and is a quotient of a 2-uniserial space group with dimension 8. Its construction is described by Felsch [5]; the verification that it is a counterexample relied heavily on the conjugacy class algorithm of Felsch and Neubüser [6], which is also used in our computations.

In this paper we discuss the class-breadth conjecture translated to p-uniserial space groups. The coclass of a p-uniserial space group S with lower central series $S = \gamma_1(S) > \gamma_2(S) > \ldots$ is defined by $cc(S) = \lim_{i\to\infty} cc(S/\gamma_i(S))$. By analogy, we define the cobreadth of S by $cb(S) = \lim_{i\to\infty} cb(S/\gamma_i(S))$. In Section 2 we recall that S has finite coclass, prove that S has finite cobreadth and obtain an explicit upper bound. Hence we can formulate the class-breadth conjecture explicitly for a p-uniserial space group S: namely $cb(S) \leq cc(S) + 1$. If S is a counter-example, then, for large enough i, it follows that $cb(S/\gamma_i(S)) = cb(S) > cc(S) + 1 = cc(S/\gamma_i(S)) + 1$, and so we obtain an infinite family of counter-examples to the original conjecture. Lower bounds for i can be obtained from the proof of Coclass Theorem A [11, Section 6.4].

Combining our explicit upper bound with the algorithm in [4], we obtain an effective algorithm for constructing all p-uniserial space groups in a given dimension which might be counter-examples to the conjecture. It is described in Section 3. We implemented this algorithm in GAP [9]. It and MAGMA [1] were used extensively in our investigations.

As an application of our algorithm, we revisited the class-breadth conjecture for the prime 2. We show that there is exactly one counter-example to the conjecture among the 4-dimensional space groups; it has a quotient of order 2^{19}, coclass 4 and cobreadth 6. We have found 64 counter-examples among space groups in dimension 8. More detail is given in Section 4.

We investigated whether a similar approach might yield an odd order counter-example. We have found none. In Section 5 we prove that all 3-uniserial space groups with dimension at most 54 satisfy the class-breadth conjecture.

Leedham-Green *et al.* [12] also studied the corresponding question for Lie algebras. In particular they exhibited a nilpotent Lie algebra of dimension 8 over GF(2) which has coclass 1 and cobreadth 3. In Section 6, we exploit recent work [2] on Lie algebras with coclass 1 to exhibit, for every finite field F, a nilpotent Lie algebra L over F with $cb(L) > cc(L) + 1$.

2 Uniserial space groups

We first recall some basic concepts related to p-uniserial space groups. More precise formulations and references for these statements can be found in [11, Chapters 4, 10].

A *space group* S is an extension of a free abelian group T by a finite group P acting faithfully on T. The group T is the *translation subgroup* and P is the *point group* of S. If T has rank m, then S is a space group of dimension m. Since P acts faithfully on T, it follows that P embeds in the general linear group $\mathrm{GL}(m, \mathbb{Z})$.

We say that S is *p-uniserial* if its point group P is a p-group and the series defined by $T_0 := T$ and $T_{i+1} := [T_i, S]$ for $i \in \mathbb{N}$ satisfies $[T_i : T_{i+1}] = p$ for all i.

The s-fold wreath product $W(s,p)$ of cyclic groups with order p has an integral representation in dimension $m = p^{s-1}(p-1)$. The standard $W(s,p)$-lattice is $\mathbb{Z}^m$ which we denote by M. Let $M_0 := M$ and $M_{i+1} := [M_i, W(s,p)]$; the series $M_0 > M_1 > \dots$ satisfies $[M_i : M_{i+1}] = p$. Every p-subgroup of $\mathrm{GL}(m, \mathbb{Z})$ is conjugate in $\mathrm{GL}(m, \mathbb{Q})$ to a subgroup of $W(s,p)$. A subgroup P of $W(s,p)$ is *p-uniserial* if $M_{i+1} = [M_i, P]$ for all $i \in \mathbb{N}$. Clearly a space group is p-uniserial if and only if its point group is p-uniserial.

The actions of $W(s,p)$ on $M_0, \dots, M_{m-1}$ yield a complete, but redundant, set of representatives for the $\mathrm{GL}(m, \mathbb{Z})$-classes of maximal p-subgroups in $\mathrm{GL}(m, \mathbb{Z})$.

2.1 The coclass of a space group

We first recall the well-known result that a p-uniserial space group has finite coclass.

Lemma 2.1 *Let S be a p-uniserial space group with point group P. Then the quotients $S/\gamma_j(S)$ for $j > c(P)$ form a series of finite p-groups with the same coclass.*

PROOF: Let T be the translation subgroup of S and let P have order p^n and nilpotency class $k-1$. Then $\gamma_k(S) \leq T$ and thus $\gamma_k(S) = T_i$ for some i. As S acts uniserially on T, it follows that $\gamma_{k+j}(S) = T_{i+j}$ for all j. Hence the finite quotients $S/\gamma_j(S)$ for $j \geq k$ form a series of p-groups with coclass $n + i - k + 1$. •

Leedham-Green, McKay and Plesken (see [11, 10.5.12] for details) proved that

$$s \leq cc(S) \leq \log_p |P|$$

for a p-uniserial space group S with dimension $p^{s-1}(p-1)$ and point group P. This upper bound is sharp: for every p-uniserial point group P, the split extension $M \rtimes P$ with its standard lattice $M = \mathbb{Z}^m$ is a p-uniserial space group of coclass precisely $\log_p |P|$. The lower bound, also sharp for some p-uniserial point groups P, is much more difficult to obtain. The algorithm of [4] can be used to determine the smallest coclass of a p-uniserial space group with given point group.

2.2 The cobreadth of a space group

The cobreadths of the $S/\gamma_i(S)$ form a non-decreasing sequence of integers (see, for example, [14]). We prove a lemma which allows us to bound the cobreadths of these quotients.

If G is a group with order p^n and $g \in G$ has a conjugacy class of size p^b, then the cobreadth of g is $cb(g) := n - b$. Thus $cb(G) = \min\{cb(g) \mid g \in G\}$.

Lemma 2.2 *Let g be an element of a p-uniserial space group S. Let $C_{S/T}(gT)$ have order p^r. Let $\overline{g} \in \mathrm{GL}(m,\mathbb{Z})$ denote the action of g on T. If $1-\overline{g}$ has x elementary divisors p and y elementary divisors 0, then*

$$cb(gT^{p^l}) \leq x + ly + r \text{ for every } l \in \mathbb{N}.$$

PROOF: Let $q = p^l$. Since the irreducible integral representations of a cyclic p-group are trivial or p-uniserial, the elementary divisors of $1-\overline{g}$ are $1, p$ or 0. Let U be the matrix for $1-\overline{g}$ and let $D = AUB$ be the Smith normal form of U. If $tU \in T^k$ then $tA^{-1}D \in T^k$ for all $k \in \mathbb{N}$ and $t \in T$, and conversely. Hence the intersection of the centraliser of gT^q with T/T^q has order p^{x+ly} if $l > 0$ (otherwise this is only an upper bound). Therefore $cb(gT^q) \leq x + ly + r$. •

Lemma 2.2 can now be applied to obtain an upper bound for the cobreadth of S. Following [11] we let $d_0, \ldots, d_{s-1}$ be the natural generating set for the wreath product $W(s,p)$ and define $c_0 = d_0 \cdots d_{s-1}$. Then c_0 is an element with order p^s in $W(s,p)$. For $1 \leq i \leq s-1$ we define $c_i = c_0^{p^i}$ and $W_i(s,p) = C_{W(s,p)}(c_i)$. For example, in $W(3,2)$,

$$d_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad d_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$I - c_0 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad I - c_0^2 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and these matrices have elementary divisors $1,1,1,2$ and $1,1,2,2$, respectively.

Theorem 2.3 *Let S be a p-uniserial space group of dimension $p^{s-1}(p-1)$ with point group P. If $c_i \in P$ for some i, then*

$$cb(S) \leq (s-i+1)p^i + p^{i-1} + p^{i-2} + \ldots + p + 1.$$

PROOF: It is straight-forward to write down the matrices $1 - c_i$ and determine that they have $(p-1)p^{s-1} - p^i$ elementary divisors 1 and p^i elementary divisors p. It follows from Lemma 2.2 that $cb(S) \leq p^i + |W_i(s,p)|$.

As observed in [4], $W_i(s,p)$ is the permutational wreath product of a cyclic group of order p^{s-i} with $W(i,p)$. Thus $|W_i(s,p)| = |W(i,p)|(p^{s-i})^{p^i}$ and hence

$$\log_p |W_i(s,p)| = p^{i-1} + p^{i-2} + \ldots + p + 1 + (s-i)p^i.$$

The result follows. •

For all S, it is easy to deduce that $c_{s-1} \in P$ [4, Theorem 19]. Hence we obtain the following.

Corollary 2.4 *A p-uniserial space group has finite cobreadth.*

2.3 Covered space groups

Let S be a p-uniserial space group with point group P and translation subgroup T. The centralisers $C_i = C_S(T_i/T_{i+2})$ are the *two-step centralisers* of S. We say that S is *covered* if $S = \cup_{i\in\mathbb{N}} C_i$. Similarly, the two-step centralisers of $W(s,p)$ are defined as $C_i^* = C_{W(s,p)}(M_i/M_{i+2})$. A p-uniserial subgroup P of $W(s,p)$ is *covered* if $P = \cup_{i\in\mathbb{N}_0}(P \cap C_i^*)$ or, equivalently, $P \subseteq \cup_{i\in\mathbb{N}_0} C_i^*$. A p-uniserial space group is covered if and only if its point group is covered.

Lemma 2.5 *If a p-uniserial space group S is not covered, then S satisfies the class-breadth conjecture.*

PROOF: If S is not covered, then $S/\gamma_j(S)$ is not covered for all j. Lemma 3.1 of [12] implies that $S/\gamma_j(S)$ satisfies the conjecture. •

Covered p-uniserial space groups occur first in dimension 4 for $p = 2$ and in dimension 54 for $p = 3$ because $W(s,p)$ has exactly s two-step centralisers and a p-uniserial point group needs at least $p+1$ two-step centralisers to cover it (see Section 4.2 of [11]).

3 An algorithm to decide the conjecture

Underpinning our algorithm is that of [4]. For odd p that algorithm enumerates or constructs, without repetition, all uniserial p-adic space groups in dimension m having coclass at most r for given positive integers m, r. This is equivalent to constructing integral p-uniserial space groups. For the prime 2 it constructs all 2-uniserial integral space groups, but the resulting list of groups may contain duplicates.

If P is a p-uniserial subgroup in $W(s,p)$ and $q := p^s/|Z(P)|$, then the actions of P on $M_0, \ldots, M_{q-1}$ describe a complete set of $\mathrm{GL}(m,\mathbb{Z})$-representatives for the $\mathrm{GL}(m,\mathbb{Q})$-class of P. Every space group S with point group conjugate to P can be obtained as an extension of M_i by P for some $i \in \{0, \ldots, q-1\}$.

Lemma 3.1 *Let S be an extension of a lattice M_i by a p-uniserial subgroup P of $W(s,p)$ for some $i \in \mathbb{N}$ and let $R = \{g \in P \mid \det(1-g) \neq 0\}$. Then*

$$cb(S) \leq \min_{g\in R}(\log_p(\det(1-g)) + \log_p(C_P(g))).$$

PROOF: This follows from Lemma 2.2. The set R is non-empty because it contains a conjugate of c_{s-1}. Since $\det(1-g) \neq 0$, the elementary divisors of $1-g$ are 1 or p and so $\det(1-g) = p^x$ where $1-g$ has exactly x elementary divisors p. •

The bound obtained in Lemma 3.1 is *independent* of the lattice M_i since conjugacy in $\mathrm{GL}(m,\mathbb{Q})$ does not change the determinant. It can be computed directly from P without constructing any extensions.

We next outline an effective algorithm which determines, for given prime p, dimension $m := p^{s-1}(p-1)$, and bound j, all p-uniserial space groups S in dimension m such that $S/\gamma_j(S)$ is a counter-example to the class-breadth conjecture.

(1) Determine up to conjugacy in $\mathrm{GL}(m, \mathbb{Q})$ the p-uniserial covered point groups in $W(s, p)$.

(2) For every such point group P:

   (a) Use Lemma 3.1 to determine an upper bound u for the cobreadths of the associated space groups.

   (b) For each lattice $M_0, \ldots, M_{q-1}$ with $q = p^s/|Z(P)|$ do:

      (i) Construct all extensions S of M_i by P of coclass at most $u-2$.

      (ii) Decide which extensions S satisfy $cb(S/\gamma_j(S)) > cc(S/\gamma_j(S)) + 1$.

The list produced by this algorithm may contain duplicates for $p = 2$.

In Step (1), we compute $W(s, p)$ and its two-step centralisers, and then construct, up to conjugacy in $\mathrm{GL}(m, \mathbb{Q})$, all subgroups of $W(s, p)$ which are contained in the union of the two-step centralisers. Part (i) of Step (2b) is reduced to a finite cohomology computation as outlined in [4, Theorem 30]. It is important to determine only the space groups of coclass at most $u-2$ as this significantly reduces the number of groups constructed. Part (ii) uses standard algorithms for p-groups.

4 Groups with 2-power order

We investigated in more detail covered 2-uniserial space groups. These have dimension at least 4.

4.1 Dimension 4

There is just one covered 2-uniserial point group P with dimension 4. It has order 64. As shown in [8], there are 8 covered 2-uniserial space groups with dimension 4. Exactly one of these is a counter-example to the class-breadth conjecture.

Lemma 4.1 *The class* 15 *quotient of* $G = \langle a, b \mid a^4, b^4, [b, a, a], [b^2, a]^2 \rangle$ *has order* 2^{19}, *coclass* 4 *and cobreadth* 6.

It can be checked that G is a 2-uniserial space group. Among the descendants (see [13]) of the class 5 quotient of G, there are 40 groups with order 2^{19}, coclass 4 and cobreadth 6.

This raises a natural question, posed by the referee. The coclass of the descendants of a *settled* p-group [11, Definition 5.4.1] is fixed. Let $G = S/\gamma_j(S)$; do descendants of order dividing $|G|$ of settled factor groups of G have cobreadth at most $cb(G)$? If so, then it may provide a method to prove that these counter-examples are the smallest.

4.2 Dimension 8

We used our algorithm to construct a complete list of covered 2-uniserial space groups in dimension 8, and among these found 64 pairwise non-isomorphic covered 2-uniserial space groups S such that S/T^{16} is a counter-example to the class-breadth conjecture. We summarise our results for the 64 space groups in Table 1.

Number	Coclass	Cobreadth
4	5	7
18	5	8
24	6	8
9	6	9
9	6	10

Table 1: Cobreadth and coclass for some space groups of dimension 8

In some cases verification of the cobreadth for a quotient $Q = S/\gamma_k(S)$ is a routine computation: we simply compute the conjugacy classes of Q and read off the size of the largest one. In other cases, this approach is not feasible: there are cases where the smallest counter-example Q has order 2^{43}. In such cases, we computed the conjugacy classes in a quotient with order at most 2^{28}; we determined class representatives in that quotient with "small" centralisers; finally, we verified that the corresponding centralisers in Q are large enough.

The smallest counter-example we have found with dimension 8 has order 2^{29}, coclass 6 and cobreadth 8. It is the class 23 quotient of the (space) group

$$\langle a, b \mid (a^{-1}b)^4, b^8, [b, a, b], (b^3a^{-2}b^{-1}a^2)^2, b^{-2}a^{-2}b^3a^{-2}b^{-2}a^2ba^2\rangle \ .$$

The space group in [5] has coclass 5 and cobreadth 8.

Polycyclic presentations for the 64 space groups are available in GAP format at `www.tu-bs.de/~beick/sp.html`, as are functions to study them.

5 Groups with 3-power order

Covered 3-uniserial space groups first occur at dimension 54. There are 188 covered point groups for space groups with dimension 54 up to conjugacy under $\mathrm{GL}(54, \mathbb{Q})$; each has centre of order 3 and so has 27 different lattices. Their orders range from 3^{32} to 3^{38}. The enumeration algorithm of [4] establishes that there are 2395542 covered 3-uniserial space groups with dimension 54.

We used Theorem 30 of [4] to show that these space groups have coclass at least 12.

Every covered point group contains a conjugate of the element c_1 of $W(4, 3)$ defined in Section 2.2. A straight-forward application of Theorem 2.3 shows that the cobreadth of each space group with dimension 54 is at most 13. Thus we obtain the following.

Theorem 5.1 *The covered* 54*-dimensional* 3*-uniserial space groups satisfy the class-breadth conjecture.*

Hence, as we observed in Section 1, so do all sufficiently large quotients of these.

We applied Lemma 2.2 to obtain a sharper upper bound for the cobreadth of each space group S with dimension 54. A random search demonstrated that each covered point group P has an element g whose related matrix $1 - \overline{g}$ has Smith normal form with three elementary divisors 3 and no elementary divisors 0 and the others 1. The centraliser of g in P has order 3^5. Hence the cobreadth of S is at most $5 + 3 = 8$.

The computations showing that the coclass of each space group is at least 12 and that its cobreadth is at most 8 took about 48 hours using GAP on a Pentium IV machine.

We also considered the covered point groups of the wreath products $W(5,3)$ and $W(6,3)$. The corresponding space groups have dimension 162 and 486. Since the number of covered 3-uniserial space groups is too large to process individually, we considered instead a small sample. None is a counter-example.

6 Lie algebras

Leedham-Green *et al.* [12, 5.1 (i)] prove that every nilpotent Lie algebra over an infinite field satisfies the corresponding conjecture. We describe, for every finite field, Lie algebras with coclass 1 and cobreadth at least 3.

In [2] a process, *inflation*, is described which constructs many Lie algebras with coclass 1 over fields with positive characteristic p. In particular Proposition 6.2 of [2] describes for the finite field $\mathrm{GF}(q)$ (graded) nilpotent Lie algebras with dimension $2p^q + 2$ and coclass 1 which are covered by their 2-step centralisers. These algebras have cobreadth at least 3 as we outline in the next paragraph. There are $(q-2)!$ such algebras.

Let L be one of these algebras and $[L, L]$ its commutator subalgebra. Since the algebra is covered by its 2-step centralisers, the centralisers of all elements of $L \setminus [L, L]$ have dimensions at least 3. Further, every element in $[L, L]$ is centralised by the last two (non-trivial) homogeneous components of L and so has centraliser with dimension at least 3.

Over $\mathrm{GF}(3)$ the construction gives one algebra L with dimension 56. It is not difficult to find an example with smaller dimension. Take the class 36, dimension 37, quotient Q of L. The covering algebra [10] of Q has dimension 39. One of its dimension 38 quotients is covered by 2-step centralisers. In fact this is the smallest dimension in which covered Lie algebras over $\mathrm{GF}(3)$ with coclass 1 occur.

Lemma 6.1 *The class* 37 *quotient of*

$$\langle a, b \mid [b,a,b], [b,a,a,a,b], [b,6a], [b,5a,b,2a,b,3a], [b,5a,(b,2a,)^4, a+b],$$
$$[b,5a,(b,2a,)^{10}, a-b] \rangle \ .$$

has dimension 38*, coclass* 1 *and cobreadth* 3*.*

(Here $[x, 2a] = [x, a, a]$ and so on, and the exponents indicate the number of repetitions of the pattern.)

Moreover, given positive k, one can find examples with cobreadth at least $k + 2$. It suffices to have each subspace of the first homogeneous component L_1 occurring at least k times as a 2-step centraliser and the first k 2-step centralisers equal. We omit the details.

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Last revised March 13, 2006