The computation of π: a selective history

After my last miniature, in which I recalled some of the popular mathematical works of Charles Dodgson, John Harper kindly reminded me of a nice identity, due to Dodgson, involving the arctan function. Since this can be used to find certain expressions from which π can be computed, the topic for the present miniature naturally suggested itself. Suppose l, m and n are positive integers then it is easy to check that $(l+m+i)(l+n+i) = (2l+m+n)(l+i)+(mn-l^2-1)$. Hence, if $mn = l^2+1$, it follows that

$$\arctan\left(\frac{1}{l+m}\right) + \arctan\left(\frac{1}{l+n}\right) = \arctan\left(\frac{1}{l}\right). \tag{1}$$

We will discuss an application of this identity near the end of this miniature.

Let $s_n = \sin\left(\frac{\pi}{n}\right)$ and $t_n = \tan\left(\frac{\pi}{n}\right)$, for n an integer greater than 2. It is easy to see that $2ns_n$ is the length of the perimeter of a regular polygon with n sides inscribed in a circle with unit radius and that $2nt_n$ is the corresponding length for a regular circumscribed polygon. Hence, if $p_n = ns_n$ and $P_n = nt_n$ then $p_n < \pi < P_n$. This inequality was used by Archimedes (circa 287–212BC), together with the recursions,

$$P_{2n} = \frac{2p_nP_n}{p_n+P_n}, \qquad p_{2n} = \sqrt{P_{2n}p_n} \tag{2}$$

and the initial values $p_6 = 3$, $P_6 = 2\sqrt{3}$, to find values the values of these bounds for $n = 12, 24, 48, 96$. The last of these gives $3.141031 < \pi < 3.142715$ so that $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

A new formula due to John Wallis (1616–1703), soon became the method of choice for the calculation of π. This is derived below in Exercise 5 and follows by substituting $z = \frac{1}{2}$ in a result published later (in 1749) by Euler

$$\frac{\pi z}{\sin(\pi z)} = \left(1-\frac{z^2}{1^2}\right)^{-1}\left(1-\frac{z^2}{2^2}\right)^{-1}\left(1-\frac{z^2}{3^2}\right)^{-1}\cdots. \tag{3}$$

This formula was soon overtaken by identities such as

$$\pi = 16\arctan\left(\frac{1}{5}\right) - 4\arctan\left(\frac{1}{239}\right),$$

which follows from the identity $(5+i)^4 = 4(119+120i) = 2(1+i)(239+i)$ or by repeated use of the textbook formula for the difference or sum of two arctangents. Calculations of π since the time of John Machin (1680–1752) have mainly relied on formulae such as (3), together with the arctan series

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$$

A sad point in the history of calculations based on the arctan function was represented by the work of W. Shanks (1812–1882) who laboriously computed 707 digits of π but got the last 180 digits wrong. Approximately 100 years later, in 1961, D. Shanks and J. W. Wrench Jr used two different arctan based formulae, of which one is

$$\pi = 24\arctan\left(\frac{1}{18}\right) + 32\arctan\left(\frac{1}{57}\right) - 20\arctan\left(\frac{1}{239}\right), \tag{4}$$

to find π to 100000 decimal places together with a check on the answer. The authors speculated whether or not a million places would ever be found and discussed some exotic ways in which this might be done. It was only 12 years later that this feat was achieved, not by using a new formula, but using better hardware.

However, effective new methods have since taken over based on the use of the arithmetic–geometric mean and related to the computation of elliptic integrals. At present count more than 10^{10} digits of π have been found.

The identity of Dodgson (1) can be used to replace a term in identities such as (4) by two more rapidly converging terms. For example, write $l = 18$, $m = 13$ and $n = 25$ and it is found that

$$\arctan\left(\frac{1}{18}\right) = \arctan\left(\frac{1}{31}\right) + \arctan\left(\frac{1}{43}\right).$$

Does this really give much of an improvement? In this case, evidently not because, if the series for $\arctan(x)$ is computed to say 10^6 decimal places, then 398319 terms are needed if $x = \frac{1}{18}$; on the other hand 335262 are needed for $x = \frac{1}{31}$ and 306096 for $x = \frac{1}{43}$. Even if the computations are done in parallel the gain is only marginal. For an example where the Dodgson identity is more beneficial, see Exercise 3.

To read more about π and the computation of its value, I recommend my main reference, “Pi and the AGM” by Jonathan M. Borwein and Peter B. Borwein (John Wiley & Sons, New York) 1987.

Exercises

1. Prove the recursions (2) using standard trigonometry.
2. Starting with $p_4 = 2\sqrt{2}$ and $P_4 = 4$, calculate a sequence of bounds analogous to those of Archimedes.
3. Use Dodgson's identity (1) to obtain an improvement to the formula $\pi = 4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{3}\right)$.
4. Verify the identity $\arctan\left(\frac{1}{n}\right) = 2\arctan\left(\frac{1}{2n}\right) - -\arctan\left(\frac{1}{4n^3+3n}\right)$. For which values of n does it speed up the computation of $\arctan\left(\frac{1}{n}\right)$?
5. Prove (3) using the Weierstrass product formula and expand the Wallis formula (found by substituting $z = \frac{1}{2}$) into factors of which those in the denominator are $1\cdot3\cdot3\cdot5\cdot\cdots$.

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