# A remarkable new formula for the digits of π

Many formulae proposed for the computation of π are based on the arctan series, or an equivalent formula based on the series $-\log(1 - z) = z + z^2/2 + z^3/3 + \cdots$. The simplest of these, famous as an example of slow convergence, is

$$\pi = 2i \log(1 - i) - 2i \log(1 + i) = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \right). \tag{1}$$

In a remarkable discovery by David Bailey, Peter Borwein and Simon Plouffe (*The Mathematical Intelligencer* **19** (1997), 50–57), a formula has now been found that uses the logarithms of complex numbers with arguments $\pm\frac{\pi}{4}$, as in (1), but now with excellent convergence. What is especially interesting about the new algorithm is that it becomes possible, for the first time, to compute any specific sexadecimal (base 16) digit — and therefore any binary digit — without having to evaluate all digits before it. For example, digit number 10^9 can now be found without, at the same time, evaluating the preceeding 999999999 digits.

In the figure, drawn in the complex plane, the circle has radius $r = 2^{-1/2}$ and centre at 1. Write $w = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, an eighth root of unity, so that the eight equally spaced points on the circumference of the circle are at $1 - w^k r$, for $k = 0, 1, 2, \ldots, 7$. We would like to be able to compute $\pi = 2i \log(1 - wr) - 2i \log(1 - \overline{w}r)$.
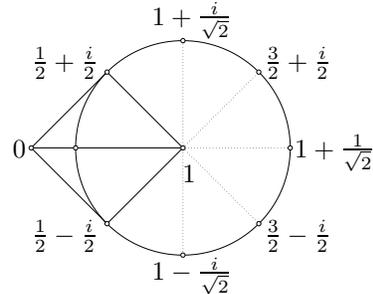
The usual series gives

$$\pi = 4i \sum_{k=1}^{\infty} r^k (\overline{w}^k - w^k)/k = \sum_{k=1}^{\infty} 4r^k \mathrm{Im}(w^k)/k.$$

Because $w^8 = 1$, the exponent in w^k can be replaced by the remainder when k is divided by 8. Furthermore, the terms for which $k \equiv 0 \bmod 4$ can be omitted because w^k is then real. This means that we can group the terms in the infinite sum into 6 separate series corresponding to $k \equiv 1 \bmod 8$, $k \equiv 2 \bmod 8$, $k \equiv 3 \bmod 8$, $k \equiv 5 \bmod 8$, $k \equiv 6 \bmod 8$ and $k \equiv 7 \bmod 8$. This gives the formula

$$\pi = \sum_{j=0}^{\infty} \left(\frac{2}{8j + 1} + \frac{2}{8j + 2} + \frac{1}{8j + 3} - \frac{\frac{1}{2}}{8j + 5} - \frac{\frac{1}{2}}{8j + 6} - \frac{\frac{1}{4}}{8j + 7} \right) \left(\frac{1}{16} \right)^j. \tag{2}$$

From the figure, we note that the line from 0 to $\frac{1}{2} + \frac{i}{2}$ is tangential to the circle. Therefore, $|\frac{1}{2} + \frac{i}{2}|^2 = (1 - \frac{1}{\sqrt{2}}) \cdot (1 + \frac{1}{\sqrt{2}})$, as we can also verify by direct evaluation. From this it follows that

$$\log \left(\tfrac{1}{2} + \tfrac{i}{2} \right) + \log \left(\tfrac{1}{2} - \tfrac{i}{2} \right) = \log \left(1 - \tfrac{1}{\sqrt{2}} \right) + \log \left(1 + \tfrac{1}{\sqrt{2}} \right)$$

or, equivalently,

$$0 = \sum_{k=1}^{\infty} r^k (-w^k - \overline{w}^k + 1 + (-1)^k)/k,$$

so that

$$0 = \sum_{j=0}^{\infty} \left(-\frac{1}{8j + 1} + \frac{1}{8j + 2} + \frac{\frac{1}{2}}{8j + 3} + \frac{1}{8j + 4} + \frac{\frac{1}{4}}{8j + 5} + \frac{\frac{1}{4}}{8j + 6} - \frac{\frac{1}{8}}{8j + 7} \right) \left(\frac{1}{16} \right)^j. \tag{3}$$

Whether or not (2) is an efficient formula for computing π, there is no doubt whatsoever that (3) is not an efficient formula for computing zero. However, we can take an arbitrary multiple of this expansion and add it to (2) to get alternative formulae for π. If the multiple is chosen as -2 we obtain the Bailey-Borwein-Plouffe formula

$$\pi = \sum_{j=0}^{\infty} \left(\frac{4}{8j + 1} - \frac{2}{8j + 4} - \frac{1}{8j + 5} - \frac{1}{8j + 6} \right) \left(\frac{1}{16} \right)^j.$$

Because the terms in the coefficient of $(\frac{1}{16})^j$ can be written as recurring sexadecimal fractions, the contributions to digit number $n > k$ can be found without needing to know the preceeding digits. For $k > n$ the terms contribute to digit n to an ever decreasing extent and only a finite number of these "guard digits" need to be evaluated.

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