A MATHEMATICAL MINIATURE

Sometimes interesting mathematics is written in the margins of books. Sometimes it is forgotten; sometimes it waits around for 300 years to reappear as one of the great pieces of mathematics of its time. Sometimes great mathematics has its origin in fleeting conversations between colleagues with different interests. Sometimes the synergy of such meetings is needed to bring all the components together.

In "Mathematical Miniatures" we will try to present some small but beautiful pieces of mathematics. Some of these will be original ideas of people who write to me and which I have managed to understand but do not recognise as already being known. Some Miniatures will be short pieces of mathematical exposition; they could for example be based on great results from classical mathematics but presented in a new and exciting manner which perhaps gives a fresh insight in the light of more modern mathematics. Some Miniatures could be teaching ideas; our subject is filled with difficult concepts and the more help we get in explaining these to students by the use of informative examples, the better. A common feature will be shortness: they will be hardly more than marginal note size. Another common feature will be that they are intended for people with a wide range of mathematical interests. We want to guarantee enjoyment and leave serendipity to chance.

To start off this new feature I will describe a simple extension to a well-known and elementary result. The extension is needed in something I have been working on. Either it is new or I have asked the wrong people. If you are the right person to correct my presumption, please write to me. Please also write to me if you have something to offer to this page of the Newsletter. Even though I will take editorial responsibility for this page, I won't make any drastic changes in any idea that is sent to me without consulting back with you. My main worry is that I will be overwhelmed with ideas for Miniatures and in this case I will have to invent some sort of procedure for deciding the order in which I present different suggestions. But for this first Miniature all I can choose from is the short note which follows.

Doubly companion matrices

Throughout this note we will denote by $p(x)$ the unitary polynomial $x^n - \alpha_1 x^{n-1} - \alpha_2 x^{n-2} - \cdots - \alpha_n$ and by $q(x)$ the unitary polynomial $x^n - \beta_1 x^{n-1} - \beta_2 x^{n-2} - \cdots - \beta_n$. Thus p and q can be characterized by the vectors $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ and $\beta = [\beta_1, \beta_2, \ldots, \beta_n]$. We define the doubly companion matrix for the pair of polynomials p and q as

$$C(\alpha,\beta) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n-1} & \alpha_n + \beta_n \\ 1 & 0 & 0 & \ldots & 0 & \beta_{n-1} \\ 0 & 1 & 0 & \ldots & 0 & \beta_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \beta_2 \\ 0 & 0 & 0 & \ldots & 1 & \beta_1 \end{bmatrix}.$$

It is well-known that the characteristic polynomial of $C(\alpha, 0)$ is p and the characteristic polynomial of $C(0, \beta)$ is q. The general result is given in this theorem.

Theorem The characteristic polynomial of $C(\alpha, \beta)$ is the polynomial given by omitting the negative powers of x in $x^{-n}p(x)q(x)$.

There really is no room for the proof of this result, for formulas for the eigenvectors or for an indication as to what earthly use there can be in doubly companion matrices. Luckily, you can stop any of these being featured in future Mathematical Miniatures by showering me with other suggestions.

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