Mathematical Apology 8

Squaring for square roots

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In Alice's Looking Glass world, it was necessary to move in what seemed to be the wrong direction to get to where you really wanted to go. Was Lewis Carroll thinking of the famous maxim in mathematics, "Man muss immer umkehren", that one should always invert? I doubt it. I have no reason to believe he knew about or was interested in elliptic functions or in Jacobi's realisation that sometimes the key to understanding a special function is to look at the inverse function. If Carroll had any mathematical meaning in mind when he constructed the looking-glass world, then it was his own idea and referred to surprises that abound in all of mathematics.

In seeking ways to calculate the square root of a number, we will look at approaches that hinge on squaring. As a starting point I quote from Mr Don Mcdonald of Wellington. If A/B is an approximation to $X = \sqrt{2}$ which is sufficiently accurate so that $|A - BX| < 1$, then $(A - BX)^2$ will be even smaller. This means that $A^2 + 2B^2 - 2ABX$ will be small so that $(A^2 + 2AB)/2AB$ will be an improved approximation to $\sqrt{2}$. Mr Mcdonald points out that this procedure can be repeated over and over again to obtain a sequence of increasingly good approximations. Here is an example of the sequence of approximations starting with $A = B = 1$, with n denoting the power to which the orginal approximation has ultimately been raised. The information in this table was supplied by Mr Mcdonald.

n	A	B	A/B
1	1	1	1.0
2	3	2	1.5
4	17	12	1.417
8	577	408	1.414215696

Actually this is another, and interesting, way of looking at the Newton method. If we seek the square root of M then the formula for the improvement on the approximation to $\sqrt{M}$ is found by squaring $A - BX$, replacing X^2 by M and solving for X. This gives the improved approximation

$$X \approx \frac{A^2 + MB^2}{2AB},$$

which can be written

$$X \approx \frac{(\frac{A}{B})^2 + M}{2(\frac{A}{B})},$$

as in Newton's method.

Now go back to the original table and insert additional values of n. For example, if $n = 3$, then A and B are found by expanding $(1 - \sqrt{2})^3 = 7 - 5\sqrt{2}$ and picking out the coefficients $A = 7$, $B = 5$. Instead of the approximation A/B, the last column now contains $A^2 - 2B^2$. The modified table is

n	A	B	A^2-2B^2
1	1	1	-1
2	3	2	1
3	7	5	-1
4	17	12	1
5	41	29	-1
6	99	70	1
7	239	169	-1
8	577	408	1

We have now found some solutions to the "Pell equation", $x^2 - 2y^2 = \pm 1$, of interest in number theory. The following puzzle was published in a newspaper in about 1915. Let m be the number of a house in a street containing exactly n houses. It is given that the totals of the house numbers less than m and greater than m are the same. That is

$$1 + 2 + \cdots + (m-2) + (m-1) = (m+1) + (m+2) + \cdots + (n-1) + n.$$

Find a solution to this Diophantine equation, where n is between 100 and 1000. The puzzle has a connection with Ramanujan because he is reported to have understood the problem and all its solutions in an instant, after hearing it stated for the first time.

Another way of using squaring to evaluate $\sqrt{2}$ is to look for the eigenvalues and eigenvectors of a matrix in terms of the powers of this matrix. For example, start with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

whose eigenvalues are $1 \pm \sqrt{2}$. If we form the successive powers of A, the eigenvectors remain the same, but the eigenvalues get replaced by their succssive powers. As this process develops, the ratio of the magnitudes of the smaller to the larger eigenvalue converges to zero. This means that A^n looks increasingly like a matrix with one of its two eigenvalues equal to zero. A consequence of this is that the columns of A^n, sc aled down to size for convenience, become good approximations to the eigenvectors of A. To get there faster, calculate only values of A^n where n is a power of 2; each of these is the square of the previous one. A value of $n = 2^4$ seems a good choice because

$$\left|(1-\sqrt{2})/(1+\sqrt{2})\right|^{16} \approx 5.6 \times 10^{-13},$$

and this small enough to be ignored to the accuracy to which we will be working.

We have

$$A^2 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 29 & 12 \\ 12 & 5 \end{bmatrix},$$

$$A^8 = \begin{bmatrix} 985 & 408 \\ 408 & 169 \end{bmatrix},$$

$$A^{16} = \begin{bmatrix} 1136689 & 470832 \\ 470832 & 195025 \end{bmatrix} \approx 1136689 \begin{bmatrix} 1 & 0.414213562373 \\ 0.414213562373 & 0.171572875254 \end{bmatrix}.$$

Our hope is that the first column of this scaled matrix, say v, will be close to an eigenvector of A. When we form Av we find

$$Av \approx \begin{bmatrix} 2.414213562373 \\ 1.000000000000 \end{bmatrix} \approx (2.414213562373)\, v.$$

Evidently the eigenvalue of greater magnitude, is approximately 2.414213562373 and we deduce a value for $\sqrt{2}$ which agrees with the correct answer to 12 decimal places.