Mathmatical Apology 7

Newton's method for computing $\sqrt{2}$

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In attempting to solve $f(x) = 0$, a good thing to try is an iteration scheme which starts with an initial approximation, x_0 (sometimes called an initial 'guess', which I don't altogether like because it doesn't sound very serious). From x_0 find a sequence of further approximations, x_1, x_2, $\ldots$, basing the value of each of these on previous approximations. In this apology, we will think only about methods in which $x_n = \phi(x_{n-1})$, for some appropriately chosen function ϕ. If the sequence of approximations is going to have any hope of converging to a zero of the function f, then we had better make sure that the set of solutions of the equation $x = \phi(x)$, that is, the set of fixed points of ϕ, includes solutions of $f(x) = 0$. The easiest way of making sure this is the case, is to construct ϕ as $\phi(x) = x - f(x)g(x)$, where g has to be decided on. We will consider some general principles about the choice of this function and then zero in, so to speak, on the calculation of $\sqrt{2}$.

When we get close to the solution we want, it would be just as well if the function ϕ is not very sensitive to small changes in x. If ξ is the fixed point we are trying to evaluate, and x_{n-1} is close to ξ, then

$$x_n - \xi = \phi(\xi + (x_{n-1} - \xi)) - \xi \approx \phi(\xi) + \phi'(\xi)(x_{n-1} - \xi) - \xi = \phi'(\xi)(x_{n-1} - \xi).$$

Thus, the value of $\phi'(\xi)$ should have a magnitude less than 1 otherwise, doing an additional iteration, once we have got close to the solution, will push it away again. Best of all we would like $\phi'(\xi) = 0$. If $\phi(x) = x - f(x)g(x)$, then $\phi'(x) = 1 - f'(x)g(x) - f(x)g'(x)$ so that $\phi'(\xi) = 1 - f'(\xi)g(\xi)$.

In the famous Newton method, g is chosen as $g(x) = 1/f'(x)$ since we will then have $\phi'(\xi) = 0$. If $f(x) = x^2 - 2$, this will give us

$$\phi(x) = x - \frac{x^2 - 2}{2x} = \frac{x}{2} + \frac{1}{x}.$$

To see how well this iteration scheme works, start with $x_0 = 100$, surely a very poor approximation to $\sqrt{2}$ but still, as we will see, close enough for us to get there in the end. Here are the members of the approximation sequence up to x_{10}. Also shown are the values of the approximations minus the value of $\sqrt{2}$.

x_0	100.00000000000000	$x_0 - \sqrt{2}$	98.58578643762691
x_1	50.01000000000000	$x_1 - \sqrt{2}$	48.59578643762691
x_2	25.02499600079984	$x_2 - \sqrt{2}$	23.61078243842674
x_3	12.55245804674590	$x_3 - \sqrt{2}$	11.13824448437281
x_4	6.35589469493114	$x_4 - \sqrt{2}$	4.94168113255805
x_5	3.33528160928043	$x_5 - \sqrt{2}$	1.92106804690734
x_6	1.96746556223115	$x_6 - \sqrt{2}$	0.55325199985805
x_7	1.49200088968972	$x_7 - \sqrt{2}$	0.07778732731663
x_8	1.41624133203894	$x_8 - \sqrt{2}$	0.00202776966585
x_9	1.41421501405005	$x_9 - \sqrt{2}$	0.00000145167696
x_{10}	1.41421356237384	$x_{10} - \sqrt{2}$	0.00000000000074

When we are a long way from the correct answer, the approximations, and therefore the errors, decrease by a factor of about $\frac{1}{2}$ in each iteration; but when we get close to the answer, quadratic convergence takes over; that is the error in each iteration is more or less proportional to the square of the error in the previous iteration.

An interesting fact about this Newton iteration scheme, if we interpret it in the complex plane rather than on just the real line, is that the sign of the real part of each approximation is the same as in the previous approximation. If we start with x_0 in the positive half of the complex plane, then we stay there and eventually converge to $\sqrt{2}$. On the other hand, if we start in the left half-plane then we stay there forever and converge to $-\sqrt{2}$. What happens if we start on the imaginary axis? We stay on the imaginary axis forever and wander up and down, never really making up our minds what we want to do. Clearly we cannot converge to anything because ϕ has only two fixed points, both real, and we cannot get near either of them. However, if we start at either of $\pm i\sqrt{6}/3$ then we alternate forever between these two values. But even this is a fragile arrangement: the slightest error in the magnitude of the starting imaginary number will push us away even from this relatively simple outcome.