

Mathematical Apology 6

A discourse on the square root of 2

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Many years ago I received a letter asking me how the square root of two might be calculated. In my excitement I replied with twenty or so ways of doing this, rather than with one good way; I have little doubt that I put the enquirer off rather than added to her store of useful knowledge. Each of the methods I proposed had some interesting mathematics hanging on to it and I now wish I had retained a copy of my reply. After more than 30 years since my original correspondence, I would like to share some methods of calculating square roots with other people. I announce, as a competition, that I would welcome learning about new and interesting ways of calculating square roots. There is no prize except an outside chance that I will feature your method in a future Apology, especially if I had not already thought of it myself or come across it somewhere else.

It is natural to shun complication and I am sure that many people in ancient days couldn't bear the thought that any numbers more esoteric than rational numbers had any right to exist. Such a person was Peripheras who set himself up as a scholar and teacher in Athens and preached an old kind of new mathematics based on purity and simplicity. He believed in the square root of two because a simple geometric construction enables a line of this length to be found, starting from a line of length one. Plato never recorded the meeting between Peripheras and Socrates or, if he did, the record has been lost. It may even have been suppressed by the followers of Euclid, who might not have wanted the credit for one of his famous arguments to be shared. However, not enough is appreciated about the interest that Socrates had in mathematics and I intend to make up for Plato's omission now by outlining the substance of what Socrates and Peripheras said to each other.

Socrates had been delayed in his arrival at a symposium at the home of Xenophon at which Peripheras was the guest of honour. Evidently some philosophical distraction had left Socrates scribbling in the sand while the party was getting into full swing. When Socrates finally arrived, Peripheras was expounding his views on education and on mathematics education in particular. The central theme was that everything had to be kept simple, not only because young minds couldn't handle sophisticated ideas, but also because mathematics itself was intrinsically simple.

"I find your ideas very interesting," quoth Socrates, "because, like you, I regard the instruction of the young as a vital matter for the welfare of the state. But surely, we must face up to complexity when we are forced to admit its necessity."

"I am honoured that you, most wise and venerable Socrates, should comment on my ideas on the teaching of arithmetic," replied his protagonist. "But," he continued, "there is no complexity about numbers save the certainty, which all rational people accept, that numbers themselves are rational." He blushed at his unintended play on words.

"Well my young friend, would you say that the square root of two is a number?"

"Certainly."

"Let me understand you clearly. Would this mean that the square root of two can be found by dividing some whole number, let us call this number 'numeratas, by another whole number 'denominatas, if only we knew what these numbers were?"

“Since all numbers can be found in such a manner and the square root of two is no exception, this would clearly follow.”

Peripheras smiled, not only at Socrates, but also to the other admiring guests, as he made this confident assertion.

“I wonder if numeratas and denominatas both have to be even numbers.”

Peripheras smiled again because he had a ready answer to this diversion. “Of course not, because the two whole numbers to which you refer could then each be cancelled by two sufficiently many times to ensure that they are no longer both even.”

“In spite of his reputation as a philosopher, Socrates is a bit of a novice in arithmetic,” Peripheras thought to himself, “not knowing the rules for cancelling fractions.”

“Since numeratas divided by denominatas is the square root of two, would this mean that the square of numeratas divided by the square of denominatas would equal two?”

Peripheras was now more impressed and, after a few moments thought, was able to verify this deduction.

“And would it then not follow that numeratas squared is twice denominatas squared?”

Again Peripheras was forced to agree but he was starting to wonder where the argument was leading.

“I wonder,” continued Socrates, “whether numeratas is even or whether it is odd.”

“I suppose it could be either because we have no idea what these numbers are,” rejoined his companion.

“But if it were odd, wouldn’t this mean that the square of it is also odd?”

“In that case this would follow, but ”

“But this would mean that twice denominatas squared would be odd. This can’t be true, can it?”

“Alright, so numeratas must be even. What is the significance of this curious conclusion?”

“Perhaps you would not mind if I brought in a new number called ‘demias, equal to half of numeratas. I think that because twice denominatas squared would be equal to four times demias squared, it might follow that twice demias squared is equal to denominatas squared. Would you check this for me please?”

Peripheras was now impressed by the mental powers of the older master but, eventually, he was able to agree with the latest conclusion that Socrates had made.

“But wait,” continued Socrates as though in surprise, “wouldn’t we now be able to conclude that denominatas is even, in just the same way as we concluded the evenness of numeratas?”

“The situation is certainly similar,” admitted Peripheras, “and I can only agree that a similar deduction can be made.”

Socrates paused as though something was troubling him. Slowly the same difficulty dawned on the other guests as well and Xenophon, as the genial host that he was, felt it was his place to enunciate the conundrum that was now pressing on everyone present.

“But didn’t you say that numeratas and denominatas can be assumed not both to be even?”

Although courtesy and good humour was the order of the day, and no one said anything to Peripheras that could be taken badly, he left the party soon afterwards complaining of tiredness.

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