Mathematical Apology 4

The Euclidean algorithim, continued fractions and rational approximations

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In continuing our discussion of the approximation of irrational numbers by rational numbers we turn first to an ancient piece of mathematics, the famous Euclidean algorithm for finding the highest common factor (now known as the greatest common divisor or gcd) of two positive integers.

If two integers M_0 and M_1 are given with $M_0 > M_1 > 0$, then any divisors they have in common are also shared with $M_2 = M_0 - n_1 M_1$, where n_1 is the quotient and M_2 the remainder when M_0 is divided by M_1. Hence, $\gcd(M_0, M_1) = \gcd(M_1, M_2)$. Because $M_2 < M_1$ we can continue this procedure to obtain a sequence $M_0, M_1, M_2, \ldots, M_k, 0$, where we can go no further because "thou shalt not divide by zero". Because M_k divides exactly into M_{k-1}, $\gcd(M_{k-1}, M_k) = M_k$, and working back throught the sequence, we conclude that $\gcd(M_0, M_1) = M_k$.

Another way of writing the steps in this algorithm, so as to put more emphasis on the quotients $n_1, n_2, \ldots, n_{k-1}$ and on the ratios of success M pairs and less on the individual M values themselves is

$$\begin{aligned} x_1 &= n_1 + \frac{1}{x_2}, \\ x_2 &= n_2 + \frac{1}{x_3}, \\ \vdots & \quad \vdots \\ x_{k-1} &= n_{k-1} + \frac{1}{x_k}, \end{aligned}$$

where we have written $x_1 = M_0/M_1$, $x_2 = M_1/M_2$, ..., $x_k = M_{k-1}/M_k = n_k$. This sequence of operations can be written as a "continued fraction"

$$x_1 = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}}.$$

or more compactly as

$$x_1 = n_1 + \frac{1}{n_2+}\frac{1}{n_3+}\cdots\frac{1}{n_k}.$$

Introducing a special notation for the purpose, we can also write

$$x_1 = [n_1, n_2, n_3, \ldots, n_k].$$

It is interesting to look at the sequence of rational numbers (known as "convergents" of the continued fraction) formed by possibly taking less than k terms. That is, the sequence

$$\begin{aligned}
\frac{N_1}{D_1} &= [n_1] &&= \frac{n_1}{1}, \\
\frac{N_2}{D_2} &= [n_1, n_2] &&= n_1 + \frac{1}{n_2} &&= \frac{n_1 n_2 + 1}{n_2}, \\
\frac{N_3}{D_3} &= [n_1, n_2, n_3] &&= n_1 + \frac{1}{n_2 + \frac{1}{n_3}} &&= \frac{n_1 n_2 n_3 + n_1 + n_3}{n_2 n_3 + 1}, \\
&\vdots \\
\frac{N_k}{D_k} &= [n_1, n_2, n_3, \ldots, n_k].
\end{aligned}$$

It is convenient to create vectors formed from the numerator–denominator pairs from this sequence. Denote these by V_1, V_2, …, V_k so that

$$V_i = \begin{bmatrix} N_i \\ D_i \end{bmatrix}, \quad i = 1, 2, \ldots, k.$$

It is remarkable, and remarkably easy to verify, that these vectors are related to each other by

$$V_i = n_i V_{i-1} + V_{i-2}, \quad i = 3, 4, \ldots, k,$$

and that the 2×2 matrices formed by putting two successive members of the vector sequence together, have a determinant either $+1$ or -1 depending on parity (evenness or oddness). That is,

$$\det \begin{bmatrix} N_{i-1} & N_i \\ D_{i-1} & D_i \end{bmatrix} = (-1)^{i-1}, \quad i = 2, 3, \ldots, k. \tag{1}$$

In passing we remark that this fact enables us to solve the Diophantine equation $ax + by = 1$, where $\gcd(a, b) = 1$, and of course x and y are required, like a and b, to be integers.

However, the main aim of this Apology is to pursue our goal of approximating positive irrational numbers by rationals. Suppose that $x_1 = X$ is such a number and not, as we have been assuming up to now, a rational number. In this case the value of each n_i is defined as the integer part of x_i and the continued fraction goes on forever. Furthermore, the approximations formed from the convergents alternate between being too small and being too large and they get closer and closer to the number being approximated. This is illustrated by the chain of inequalities

$$\frac{N_1}{D_1} < \frac{N_3}{D_3} < \frac{N_5}{D_5} < \cdots < X < \cdots < \frac{N_6}{D_6} < \frac{N_4}{D_4} < \frac{N_2}{D_2}.$$

If we approximate X by N_i/D_i, then the error in the approximation is less than $|N_i/D_i - N_{i+1}/D_{i+1}|$, because X is between N_i/D_i and N_{i+1}/D_{i+1}. Hence,

$$\left|X - \frac{N_i}{D_i}\right| < \left|\frac{N_{i+1}}{D_{i+1}} - \frac{N_i}{D_i}\right| = \frac{|N_{i+1}D_i - D_{i+1}N_i|}{D_{i+1}D_i} = \frac{1}{D_{i+1}D_i},$$

where we are able to simplify the numerator because of (1). Since $D_{i+1} > D_i$, we can bound the error in the approximation by $1/D_i^2$. Hence we have a construction for an infinite sequence for "quadratically good" approximations.

To see how this works, consider the continued fraction for π. We can find the first few terms numerically if we start with an accurate enough approximation to π. The continued fraction turns out to be $[3, 7, 15, 1, 292, \ldots]$ and the steps in the calculation of these terms are shown in the following table, together with a list of the convergents and the errors in the approximations.

i	x_i	n_i	$\frac{N_i}{D_i}$	$\pi - \frac{N_i}{D_i}$
1	3.141592653589793	3	$\frac{3}{1}$	0.141592653589793
2	7.062513305931046	7	$\frac{22}{7}$	−0.001264489267350
3	15.996594406685720	15	$\frac{333}{106}$	0.000083219627529
4	1.003417231013373	1	$\frac{355}{113}$	−0.000000266764189
5	292.634591014395472	292		

Other interesting examples can be found from some situations where the continued fractions recur, such as

$$\begin{aligned} \sqrt{2} &= [1, 2, 2, 2, 2, \ldots], \\ \frac{1+\sqrt{5}}{2} &= [1, 1, 1, 1, 1, \ldots]. \end{aligned}$$

Even though this is only the fourth of my Apologies the so-called millenium may be a suitable time for taking stock. I have absolutely no idea if anyone reads this regular feature of the Magazine and, if so, if anyone find it at all interesting. I ask for your comments.