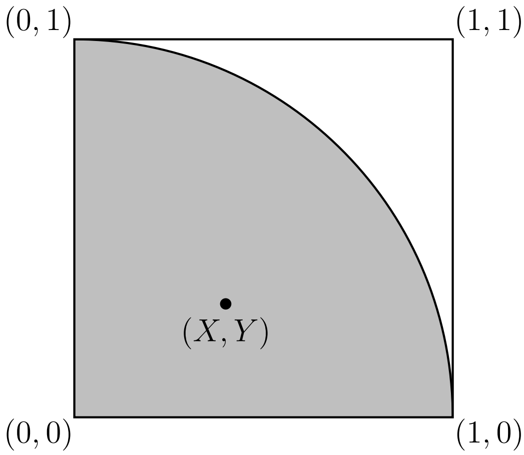
Apology number 19: $\pi \pm \sigma/\sqrt{N}$

Writing mathematics for publication is a little like trying to communicate with intelligent life somewhere in the universe. It is done with the assumption that there is someone out there who will read it and respond. Even these lightweight pieces, which I call Apologies, beg for answers. I would love to hear from somebody who loves some particular piece of mathematics so much that they would take it, alongside their favourite book and favourite piece of music, to a desert island. What is your Desert Island Formula? What is your Desert Island Algorithm? What is your Desert Island Geometric Construction? What is your Desert Island Theorem? If you are a mathematics teacher, what topic always evokes an awed and excited reaction from your class? If you are a mathematics student, what mathematical event in your life first made you realise that this is what you wanted to learn more about and even devote your life to? Not only am I receptive to hearing from you but, in these e-mail days, I am also very easy to contact.

Amongst the many outrageous ideas in science is the suggestion that it is possible to use randomness to calculate something that has nothing whatever to do with probability. The value of π, for example, can be looked at as 4 times the probability that a (uniformly distributed) random point (X, Y) in a square falls inside the inscribed circle. Because random numbers on the interval $[0, 1]$ are readily available in computers, it is more convenient to work only in a single quadrant and we have the following method of approximating π: For some integer N, select $2N$ random numbers X_i, Y_i for $i =, 2, \ldots, N$. Count how many times $X_i^2 + Y_i^2 \leq 1$. If this number is n then we can take as an approximation $\pi \approx \widehat{\pi} = 4n/N$. This is so easy to do that I tried it 10 times with $N = 1000000$. Here are the answers:

n	$\widehat{\pi}$
785511	3.142044
785875	3.143500
785307	3.141228
785553	3.142212
784868	3.139472
785340	3.141360
785812	3.143248
785018	3.140072
784664	3.138656
785696	3.142784

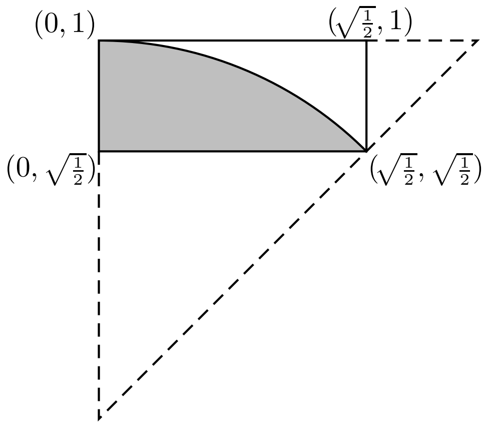
The values that X and Y can have is shown in the diagram, with the region satisfying $X^2 + Y^2 \leq 1$ shaded:



How accurate are these approximations? We can do a simple statistical analysis by estimating the mean and variance using the formulae $m = \sum \widehat{\pi}/10$, $s^2 = \sum(\widehat{\pi} - m)^2/9$. These give the values, $m = 3.1414576$, $s = 0.001627$. Using the value of m as an approximation for π gives a standard deviation $s/\sqrt{10} = 0.000514$. This might seem to be reasonable accuracy but to get just one more decimal place will take about 100 times as much computational effort.

This so-called Monte-Carlo method of calculating seems to be plagued with this handicap, that the error that should be expected goes down only as the $-\frac{1}{2}$ power of the sample size. For computations based on numerical integration we would get much more rapid improvement as the cost of the computation increases. For Simpson's rule for example, errors go down as the -4 power of the computing time. We cannot overcome the handicap in the long term, but we can reduce the variance so that we gain advantages, at least in the short term.

One way of doing this is to work in a octant, rather than a quadrant, and to remove from consideration two triangular regions, for which the area is known. What we are left with is shown in the diagram:



To approximate π we need to calculate 8 times the shaded area and add 2. Thus we choose X_i randomly in the interval $(0, \sqrt{\frac{1}{2}})$ and Y_i randomly in the interval $(\sqrt{\frac{1}{2}}, 1)$. If n is the number of samples falling in the shaded region then the approximation to π will be $\widehat{\pi} = 4(\sqrt{2} - 1)n/N + 2$. The results obtained in this experiment were

n	$\widehat{\pi}$
688892	3.141394
688777	3.141203
689023	3.141611
689086	3.141715
689292	3.142056
689284	3.142043
689061	3.141674
689122	3.141775
689121	3.141773
688467	3.140689

From these figures we obtain $m = 3.141593$ and $s = 0.000410$ suggesting that m can be relied on as an approximation to π to an accuracy of about $s/\sqrt{10} = 0.000130$. The improvement is by a factor of approximately $\frac{1}{4}$, comparable to what would be obtained in the original calculation by using a sample size 16 times as large.

Other improvements are possible but it now seems hardly worth the effort. I said that the Monte-Carlo method was an outrageous idea. So it is, but it is useful in many applications. The computation I have described is, unhappily, not one of them.