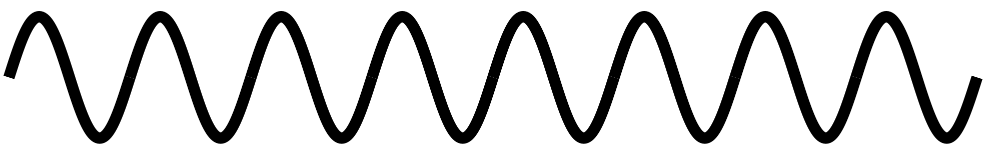
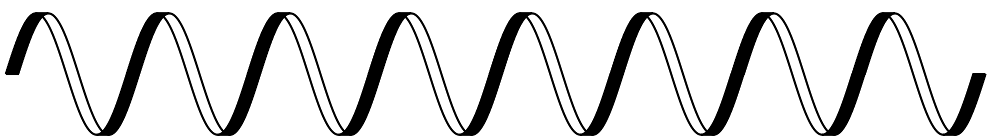
Apology October 2002

Hidden dimensions

Sometimes what might seem like complication are really simplifications. Everyone recognises the graph shown in this diagram



as denoting the sine function. Modify the way the curve is depicted and we get this picture



The sine function can now be seen for what it really is, the projection onto a plane of a function that actually evolves on the surface of a cylinder.

A famous example where three dimensional geometry is simpler than two dimensions concerns "Desargues' Theorem". We can talk about this result in a very sparse sort of geometry in which the only relevant properties are whether lines and points lie on planes or not and whether points lie on lines. Because these properties relations between points, lines and planes are not affected by projections, this system is known as "Projective Geometry". There is a slight complication with respect to parallelism but this is a little detail that can be brushed aside. For example, sets of lines in a plane that are mutually parallel can be regarded as meeting at an invented point. All these new points can be regarded as lying on a new line, the "line at infinity", also introduced to make the whole thing work properly and to transform correctly under projections.

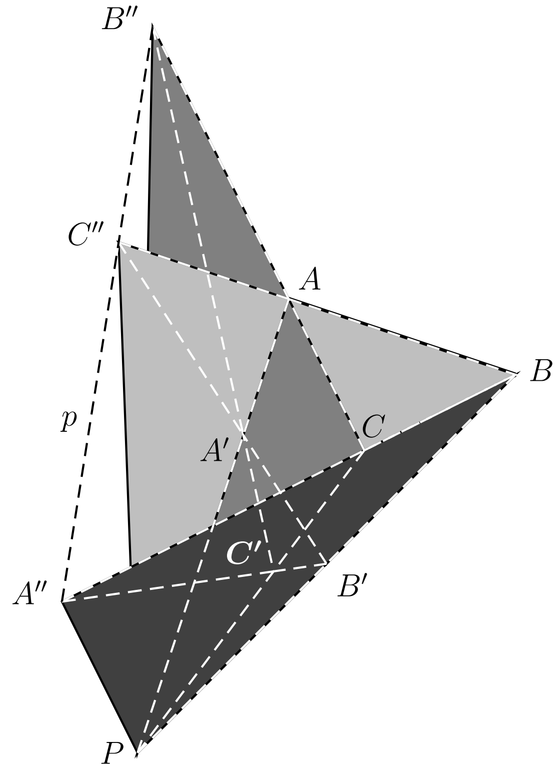
Although projective geometry was not fully formulated until the nineteenth century, the essential ideas were worked out by Girard Desargues (1591 - 1661). In particular he studied perspective with particular application to building. Desargues' theorem is actually concerned with perspective triangles and dual perpective triangles (dual in the sense of interchanging the roles of lines and points).

Suppose three plane α, β and δ meet at the point P and that two further planes δ and ϵ cut across the original three planes. Various points of intersection between various trios of the planes are denoted as in the table

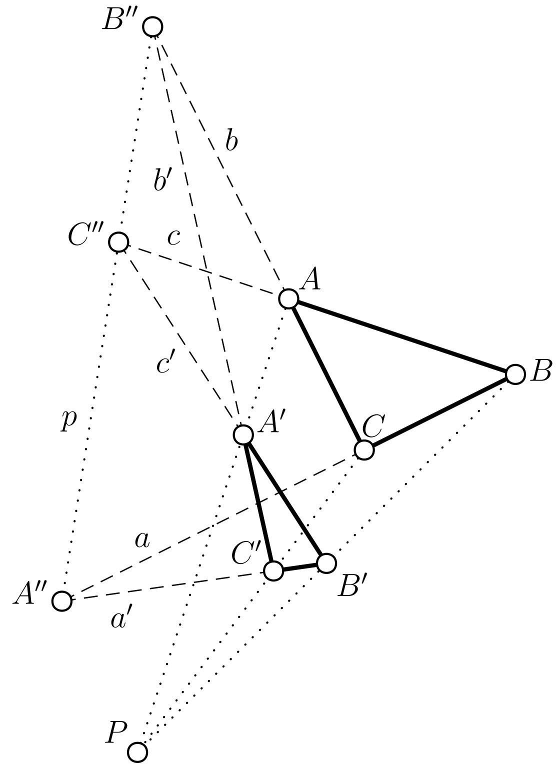
plane 1	plane 2	plane 3	point
β	γ	δ	A
γ	α	δ	B
α	β	δ	C
β	γ	ϵ	A'
γ	α	ϵ	B'
α	β	ϵ	C'
δ	ϵ	α	A''
δ	ϵ	β	B''
δ	ϵ	γ	C''

The line p is defined to be the intersection of the planes δ and ϵ. The three named points on this line, A'', B'' and C'' lie on this line.

We present two diagrams, the first showing the planes α, β and γ, with some of the other details



The second version of the diagram includes all the points and lines that have been named. The three dimensional nature of this diagram is not now really significant and it could be regarded as having been projected onto a plane.



Now this is Desargues' theorem: "Let ABC and $A'B'C'$ denote two triangles and write a, b, c denote the (extended) sides of ABC and a', b', c' the sides of $A'B'C'$. If the lines AA', BB' and CC' all pass through the same point P then the points where a and a' intersect, where b and b' intersect and where c and c' intersect, lie on the same line."

Because the result in three dimensional projective geometry follows simply from the intersections of various lines and planes, as in the diagrams, it seems to be true in purely plane geometry by simply projecting the diagram onto a plane. But the rather amazing thing is that the result is not a consequence of the axioms of incidence of points and lines in the plane alone. Other assumptions have to be made or, as we have argued it here, the plane has to be capable of being embedded in three dimensions.