Mathematical Apology 11
A beautiful picture

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One of the most despicable things a reviewer can do is to assess some work of art, literature, science or mathematics without actually sampling it. In detective fiction, absentee theatre reviews have been used as attempted alibis by reviewers who happen also to be criminals. They always seem to come unstuck by missing some crucial detail, such as the outright cancellation of the performance, thus rendering the alibi nature of their report completely useless. Thus if I were to attempt to review "A Beautiful Mind" before it has even opened in New Zealand I would put my credibility in great danger. However, there is, I hope, nothing wrong with my saying something about the work of John Nash, the subject of this biographical film.

As it happens, there was another movie with a mathematical flavour shown a few years ago. It was called "Good Will Hunting" and told the story of a young man with an aptitude for mathematics but with no training or proper education. Setting aside, as mild fantasy, Will's ability to solve problems without even knowing the language in which they were expressed, the most unbelievable thing to me was depicted in a lecture that a famous mathematician was supposed to have given. At the end, when a student asked a question about the work, the lecturer rather dismissively told the student that this sort of query would be answered by the assistant he always seemed to have with him. It was not just that he regarded such a legitimate interest as beneath his dignity to respond to in person, but that he would have an offsider available to help him out in such a way.

However, back to Nash. This mathematician was awarded a Nobel Prize in Economics for a discovery he made many years ago, before a mental illness disrupted his scientific career. The discovery is concerned with a branch of mathematics known as The Theory of Games. A game in this sense is a contest between two or more players in which each player independently makes a choice between several alternatives. The outcome of the game for each player depends on the choices made by all of the players.

A well-known example models the possible behaviours of two potentially hostile nations. During a period of border tension, nation A can choose to either make an all-out attack on nation B or else try to make peace. For nation B the choices are similar. If either makes war against the other, where the intended victim has decided to follow the path of peace, the belligerent nation wins a great victory and could be said to be better off than if both had acted peaceably. On the other hand if both A and B simultaneously decide to attack each other then there will be some loss of life, as well as a waste of resources that could have been used for the benefit of their citizens. However, neither country would achieve a victory and they would each finish up better off than if they had suffered a pre-emptive strike from the other.

How should each nation rationally behave? Nation A could reason like this: "If the enemy is going to attack me, then I am better off attacking at the same time. However, if the enemy is going to act peacefully, then I am still better off attacking. Thus, whatever the enemy decides, I should attack." The reasoning of B would be just the same and the outcome for each of them would be the misery of a war that they neither win nor lose.

This type of game differs from a "zero-sum" game in which the total of the benefits to all players is independent of the choices they make. An example of a two-person zero-sum game would be one in which each player again has two choices which we will refer to as 0 and 1. If both choose 0, or both choose 1, then each receives zero. On the other hand, if one of them chooses zero and the other chooses 1 then the person who chooses 1 obtains a score of -1 and the other receives a score of $+1$. Whatever combination of choices is made, the total amount paid out has zero total. For this zero-sum game, the optimal play for each player is to always choose 0.

For a game which does not have this zero-sum property, the closest thing to a play which is optimal for each player is what is known as a "Nash Equilibrium". The discovery of the existence of these combinations of plays and an understanding of their application to economic situations, is the work for which John Nash is most famous and for which he was awarded a Nobel prize.

Briefly, a Nash equilibrium is a set of choices by the various players in a game with the property that if any single player had made a different choice (with none of the other players making a change), then this player would be worse off. In the case of the game I have described between two potentially warring nations, the only Nash equilibrium is the one in which each player attacks the other, because if either one of the two participants makes the other choice, then this player-nation would be worse off.

It is interesting to modify the zero-sum game I have also described so that if each player chooses 1, then each receives a pay-off of x units where $x > 0$. In this case there is still only a single Nash equilibrium, in which each player still chooses 0. However, both players choosing 1 becomes increasingly attractive as the value of x is increased, as long as each player is willing to risk making this choice in the hope that the other player will also make this choice.

It is natural to think of non-zero-sum games as models for the conflict between self-interest and altruism. The choices that will lead to some sort of greater good might not be the same as the choices that might lead to a best outcome for oneself, if it is suspected that other players will act in a similar way. Personal ethical principles might be regarded as imposing a surcharge on some choices, or an additional reward for others. For example, the choice of a nation acting peaceably towards a neighbour might lead to its better treatment by other countries. On the other hand, unprovoked belligerence could lead to sanctions from the rest of the world, at worst, and general mistrust, at best.

Another interpretation of this game is as the "Prisoners' Dilemma" in which two criminal suspects are being interrogated in different rooms. Each has the choice of offering evidence against the other or refusing to do so. In the case that neither accuses the other, they are both likely to be convicted, but of a minor crime carrying a lesser penalty. If they both offer evidence against the other then the penalty would be much more severe but not as severe as it would be if only one of the two is found guilty on the evidence of the other. In this last case, the fellon who turned informer would be given a new identity in another country and other related benefits. The ethical principle, which might encourage the two suspects to act altruistically even though this would involve a risk to themselves, is the underworld code of loyalty towards fellow criminals.

I want to conclude this brief introduction to game theory, by describing a game with so many players that it is better to represent the choices continuously. Suppose that a fraction $1 - t$ of the players make some particular choice and that t make an alternative choice. For example, the choice might be between using one's own car to travel to work (number this choice as 0) and using public transport (choice 1). Let $f(t)$ denote a measure of the benefit to an individual commuter in choosing 0 given that a fraction t of the other players choose 1. Similarly $g(t)$ will denote the benefit to this player of choosing 1 in the same circumstances. We would expect both $f(t)$ and

$g(t)$ to increase as t increases, because fewer cars on the road would lead to less congestion. It might also be the case that $f(t)$ is always greater than $g(t)$; that is, driving ones own car would always be preferable to using public transport, no matter how many other cars are on the road. A final reasonable assumption is that $g(1) > f(0)$; that is, everyone would be better off using public transport than if everyone used individual vehicles.

The Nash equilibrium would be that everyone would choose 0, because anyone unilaterally making the more public-spirited choice of 1 would be worse off. Subsidising public transport at the expense of private motorists could tip the balance in the sense that the Nash equilibrium would move to everyone choosing 1 if the subsidy and compensating tax were large enough. It is an interesting paradox that, if this proposition came up for a vote, it might be logical for everyone to support it whereas, as individuals, they would act on the basis of self-interest if they could do so without penalty.

From the reports I have received about "A Beautiful Mind" from friends in the USA, the film is about the life of John Nash as much as about his fundamental contributions to game theory. We are starved of movies that express a genuine appreciation of science and mathematics, and show us something of the lives of scientists and mathematicians. From what I have heard, this is a film that should not be missed.

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