Order and stability for general linear methods

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I will discuss some of the conflicts between order and stability using order arrows and order stars to illustrate how they are inter-connected.

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General linear methods and Obrechkov methods

General linear methods are multivalue-multistage methods in which the input to a step $y^{[n-1]}$ and the output from the step $y^{[n]}$ are related to the stage values Y and the stage derivatives $F = f(Y)$ by the equations

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For the "linear test problem" $y'(x) = qy(x)$, we obtain the solution in the form

$$y^{[n]} = M(z)^n y^{[0]}, \qquad z = hq,$$

where $$M(z) = V + zB(I - zA)^{-1}U.$$

Because we are interested in stable behaviour of powers of $M(z)$, we want to know properties of the stability function $\Phi(w,z)$ given by

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The open stability region is the set of values of z in the complex plane for which any solution to the equation

$$\Phi(w, z) = 0$$

lies in the interior of the unit disc.

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Because we will not consider details of the method, but only the stability function, we will regard this as the definition of the order of $\Phi(w, z)$.

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We will look at two examples, each of which is a second-derivative generalization of a BDF method.

The first example has order 4

$$y_n = \frac{66}{85}hy'_n - \frac{18}{85}h^2y''_n + \frac{108}{85}y_{n-1} - \frac{27}{85}y_{n-2} + \frac{4}{85}y_{n-3},$$

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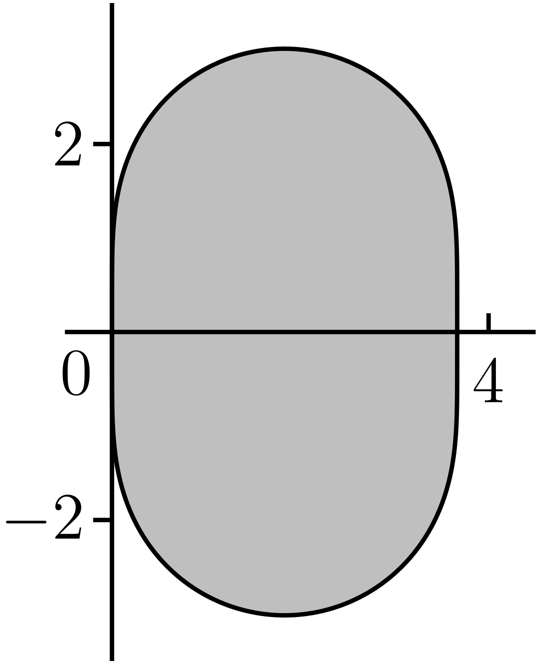
This is the $[2, 0, 0, 0, 0]$ approximation.

The stability regions of these two methods are the unshaded regions in the diagrams:



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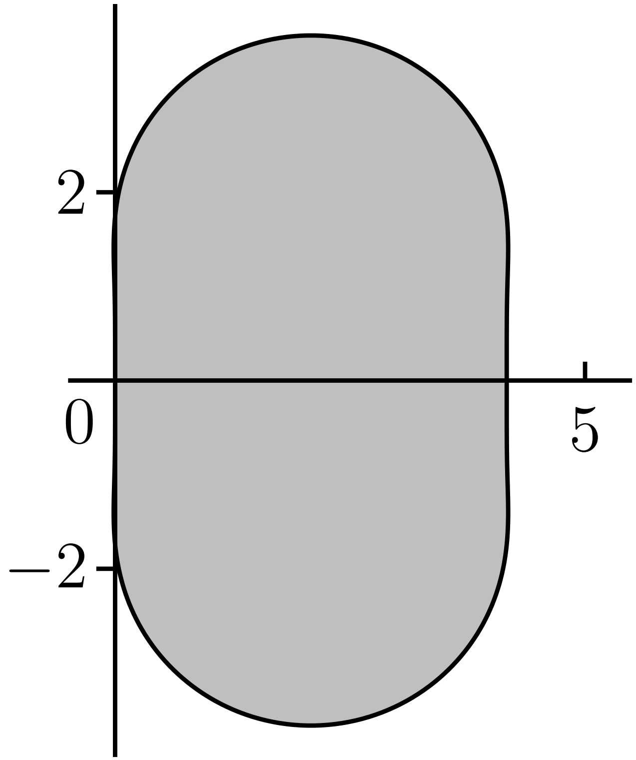

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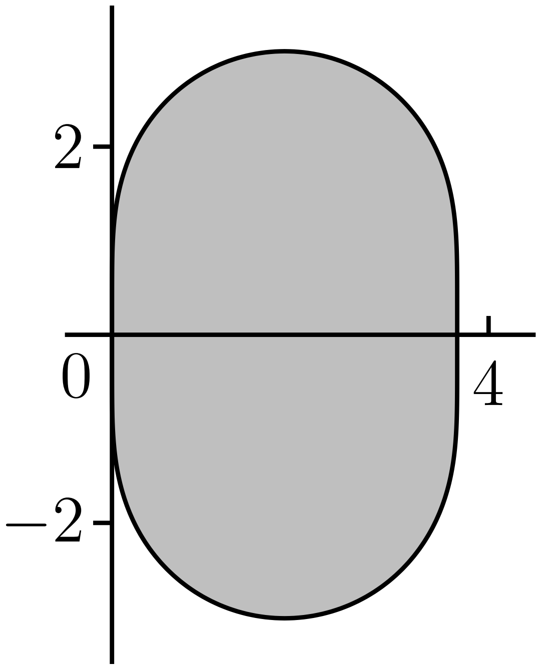

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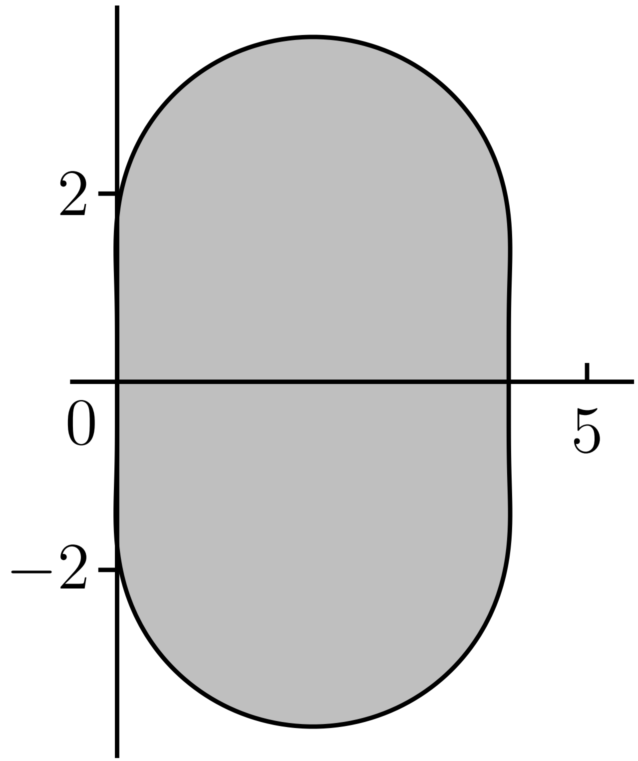

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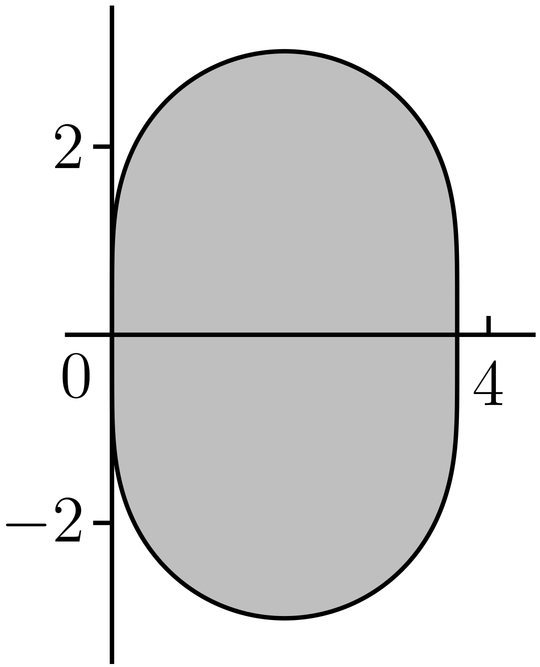

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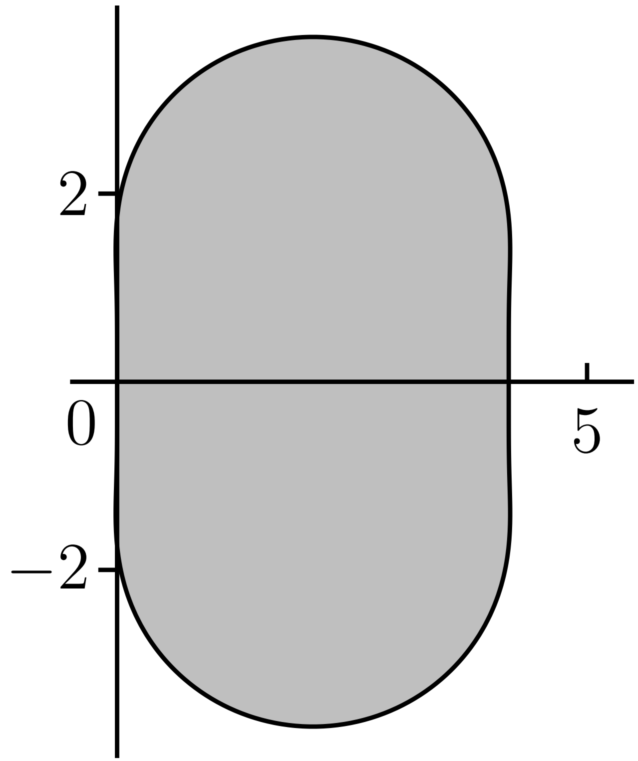

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The methods are A-stable and A(89.365°)-stable respectively.

Order stars and order arrows

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For order arrows we consider the set of (w, z) pairs satisfying $(\star)$ such that w is real and positive.

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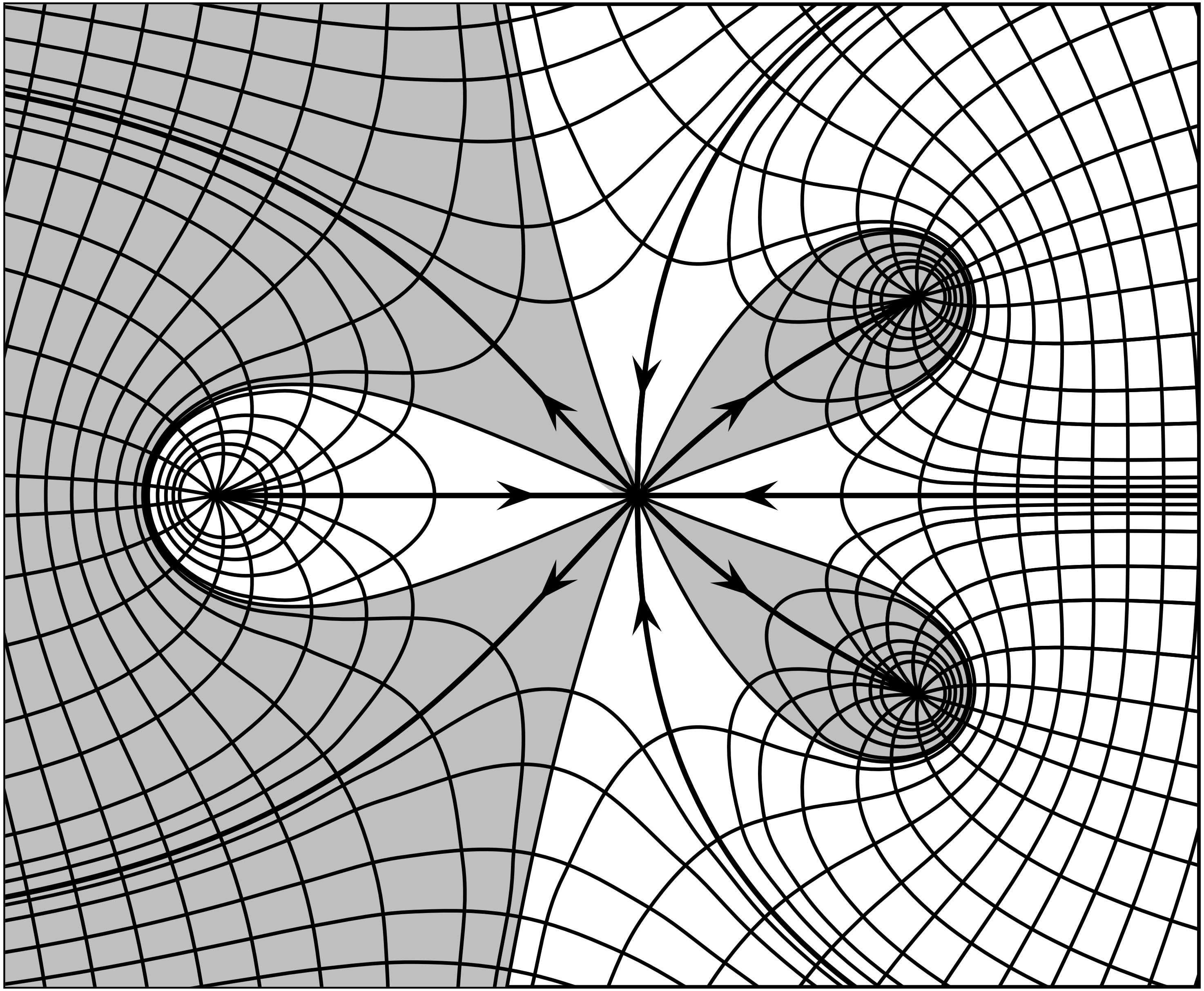
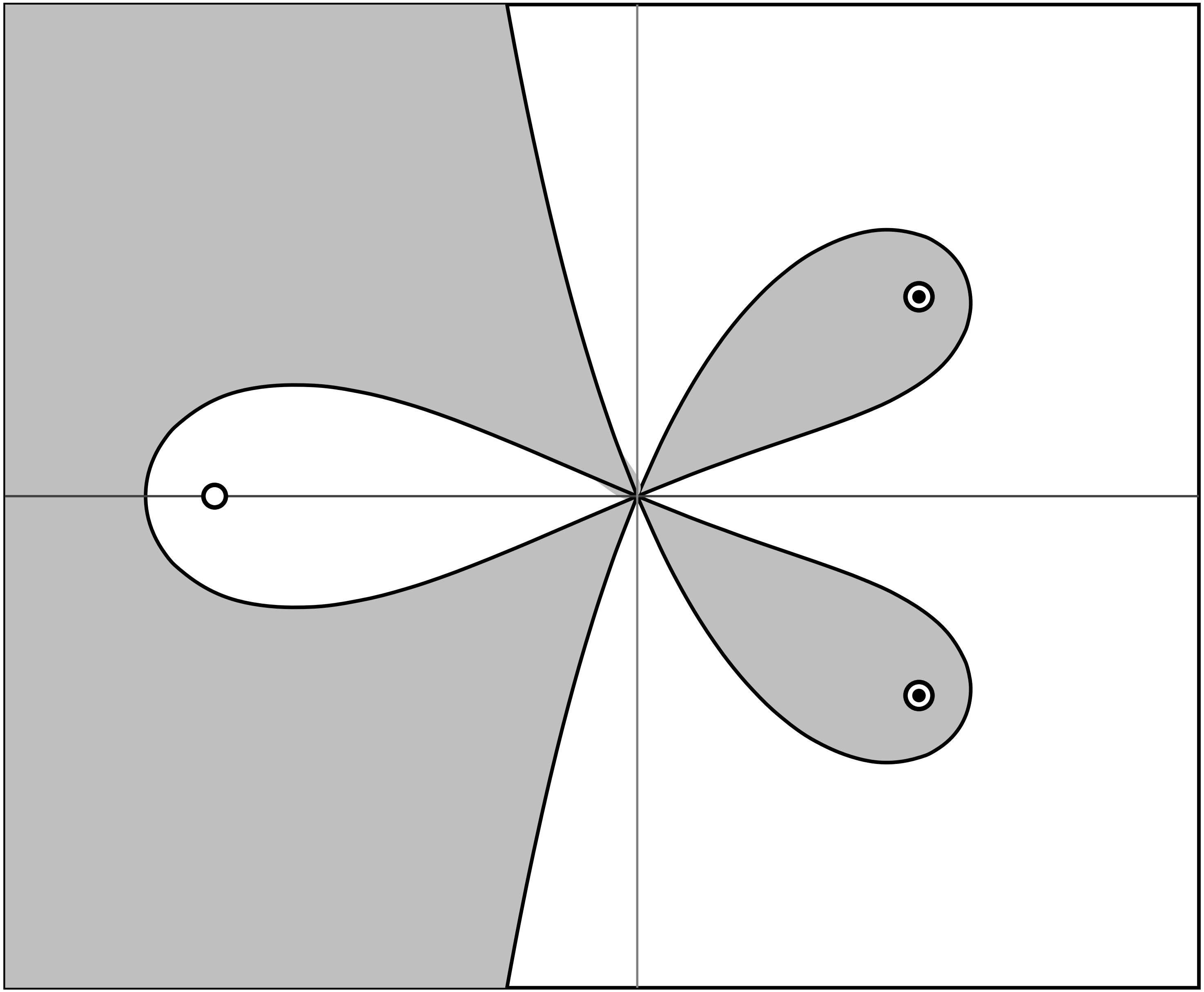
The figure on the next slide gives information on both the order star and the order arrows:



We can separate out the order star picture



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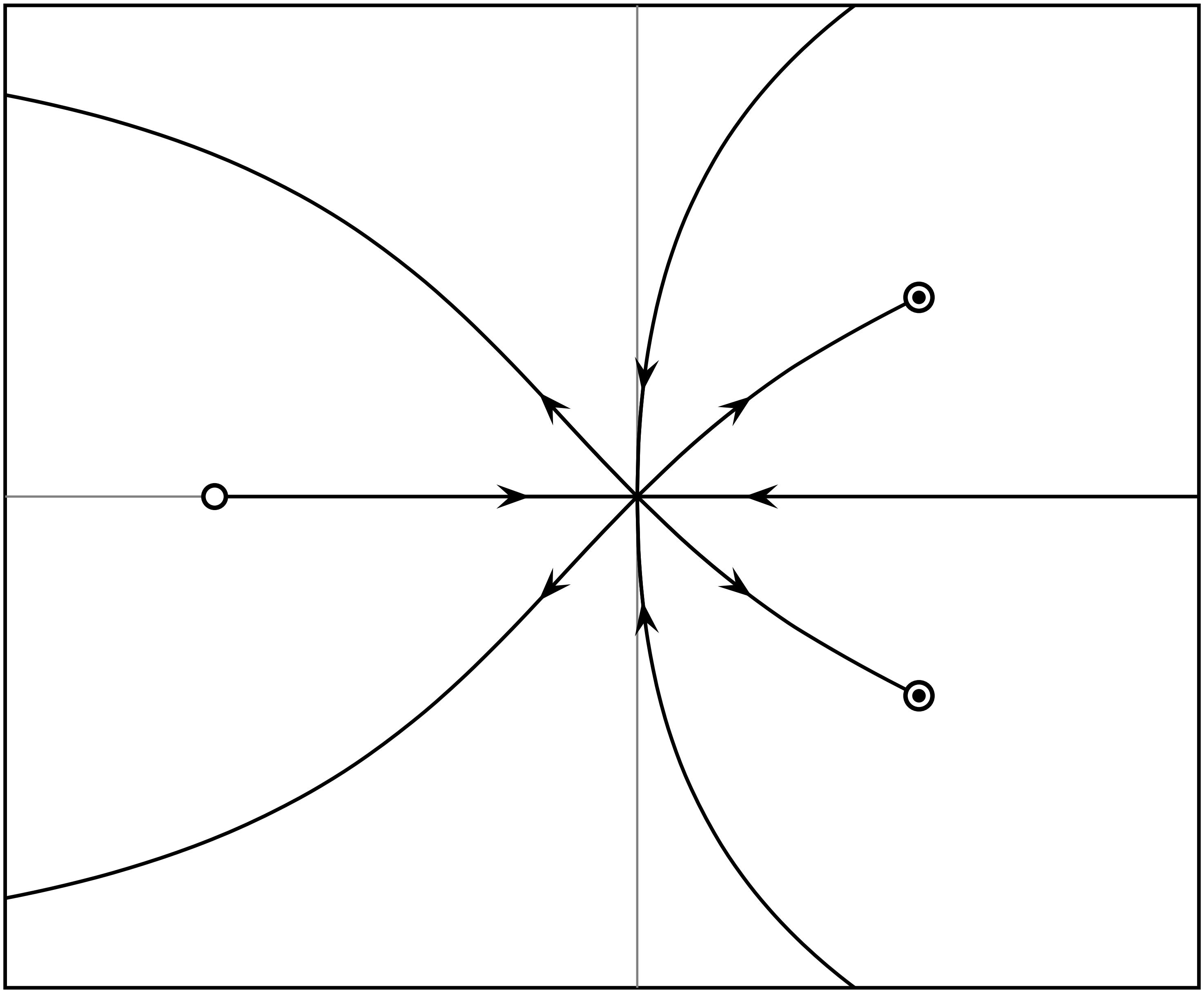
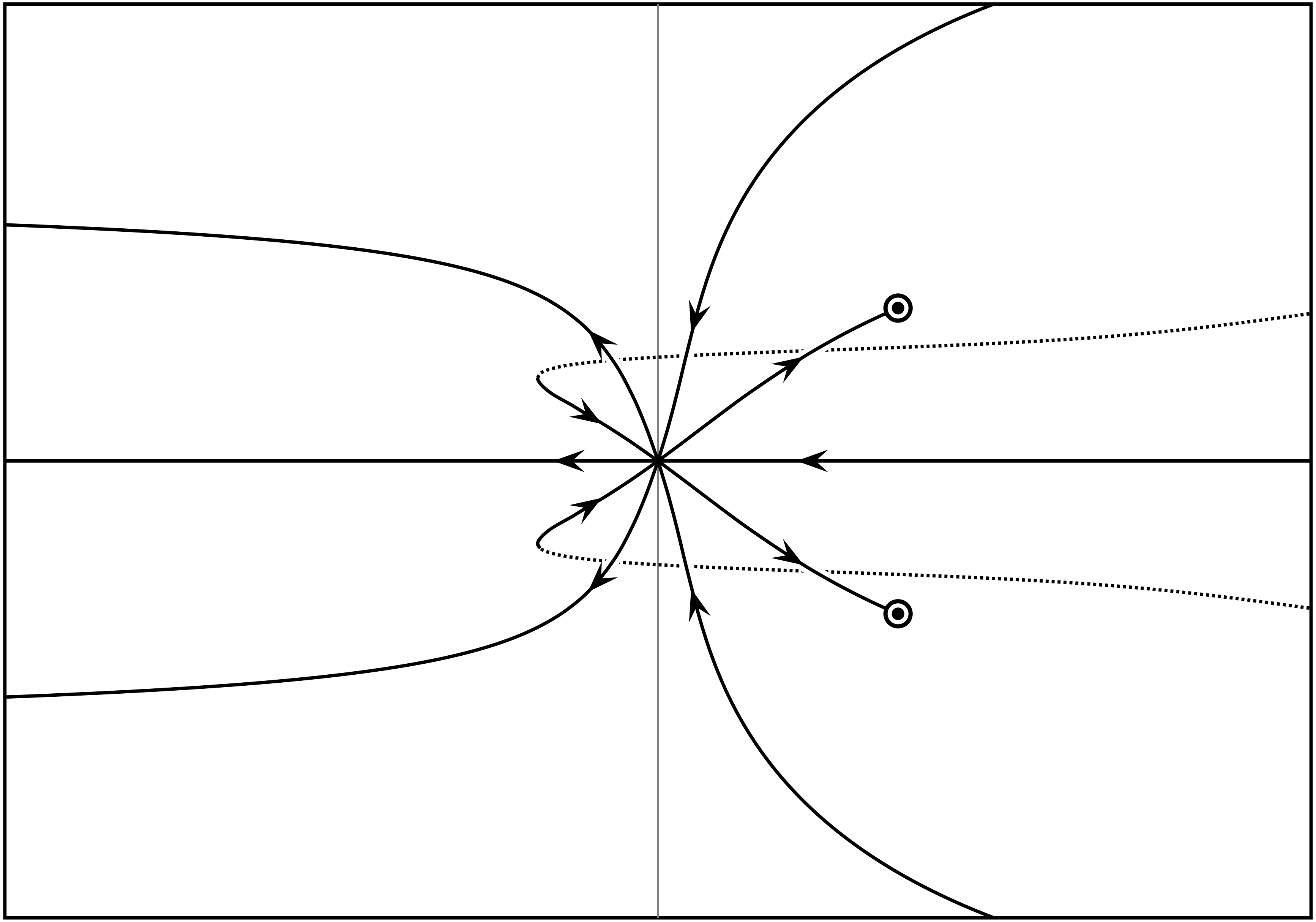


Now consider the $[2, 0, 0, 0]$ ***approximation***



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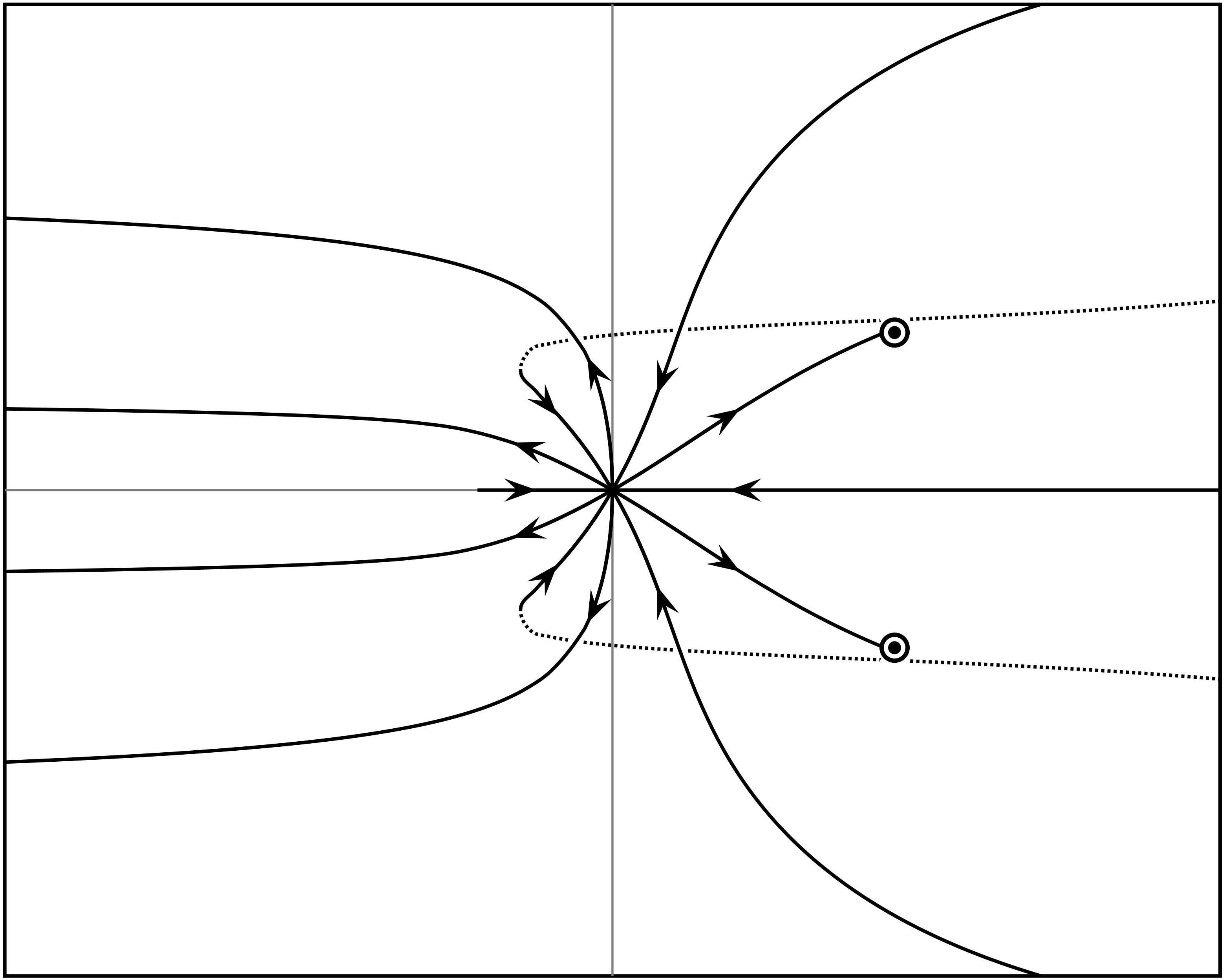

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In each case we also use the behaviour near zero of the locally defined function $w(z) = 1 + Cz^{p+1}$.

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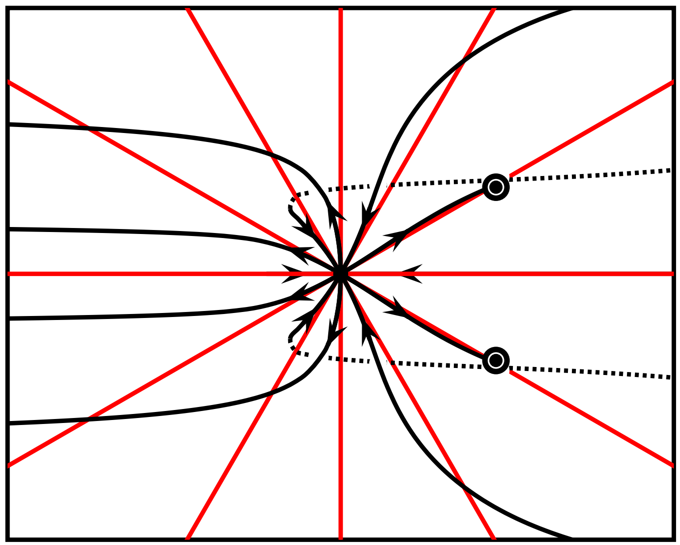

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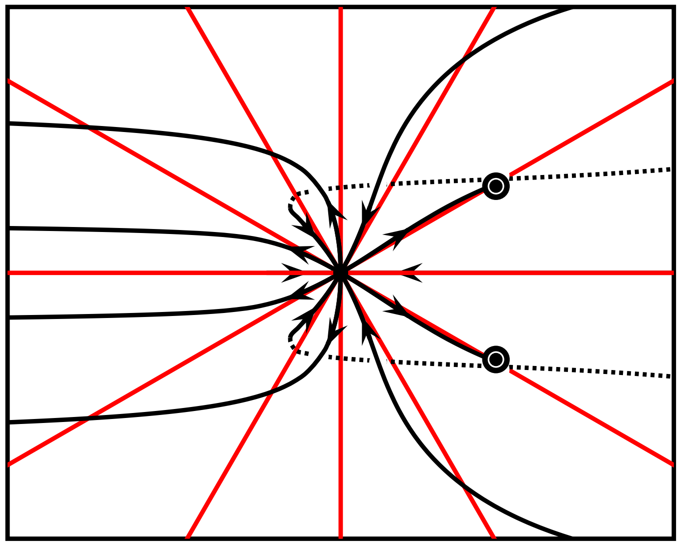

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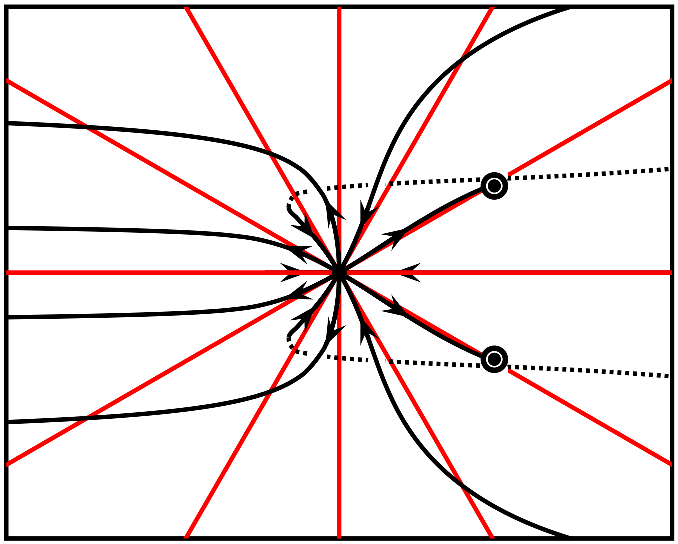

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Hence there exist up-arrows tangent to the imaginary axis.

Padé approximations to the exponential function

We consider approximations of the form

$$w = \frac{N(z)}{D(z)} = \exp(z) + O(z^{p+1}),$$

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An alternative proof will be outlined using order arrows.

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This question will be discussed later.

Because adjacent up-arrows subtend an angle

$$\frac{2\pi}{p+1}$$

and n_0 of them terminate at poles, the total angle subtended is at least

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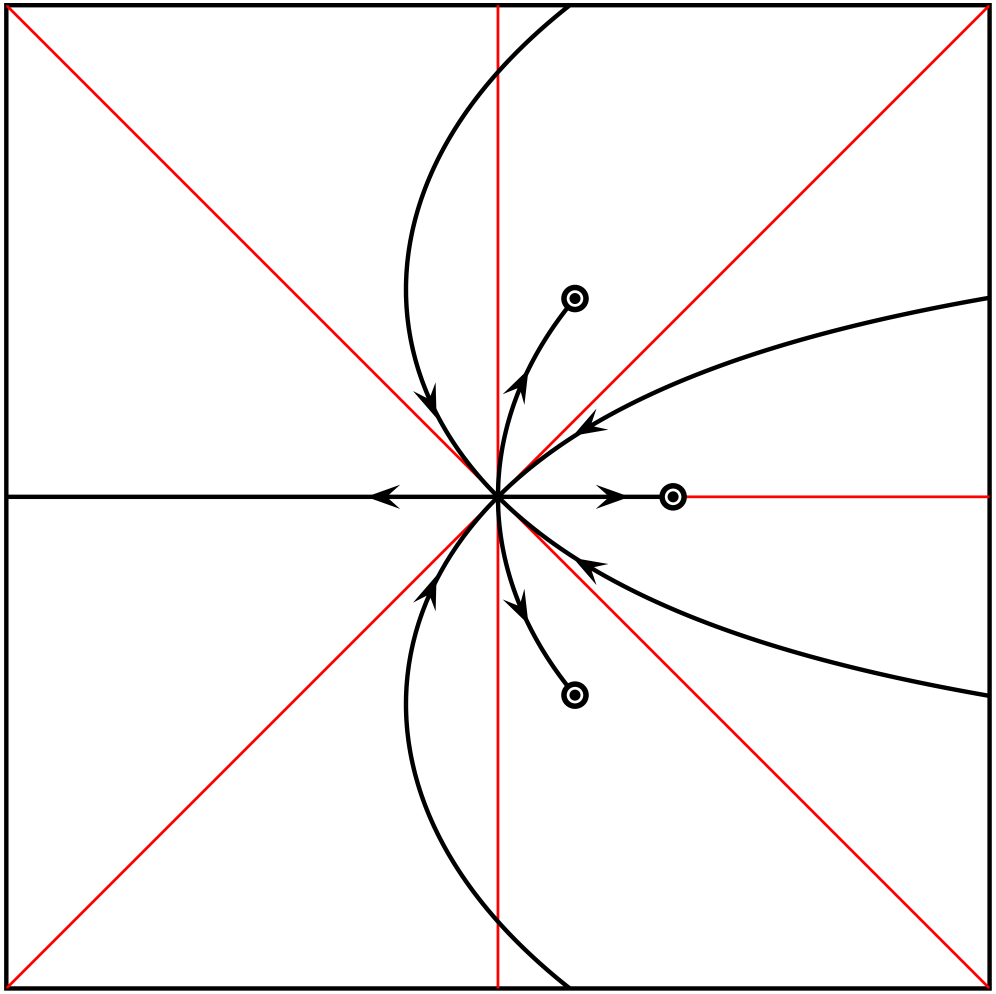
Hence, either up-arrows terminating at poles are tangential to the imaginary axis or protrude into the left half-plane.

In the latter case, there are poles in the left half-plane or an up-arrow crosses back across the imaginary axis.

We will illustrate this result in the $[3, 0]$ case.



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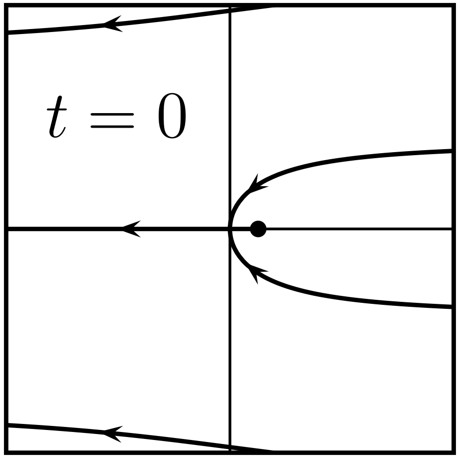
But in the general case, where everything happens on a Riemann surface, we cannot use this argument in a simple way.

We will look to see if homotopy might be a useful approach.

$$\begin{aligned}\Phi_t(w,z) &= (1-t)\Big(w(1-z)-1\Big) + t\Big(w(1-z+\tfrac{1}{2}z^2)-1\Big)\\ &= (1-t)\Phi_{[1,0]}(w,z) + t\Phi_{[2,0]}(w,z)\end{aligned}$$



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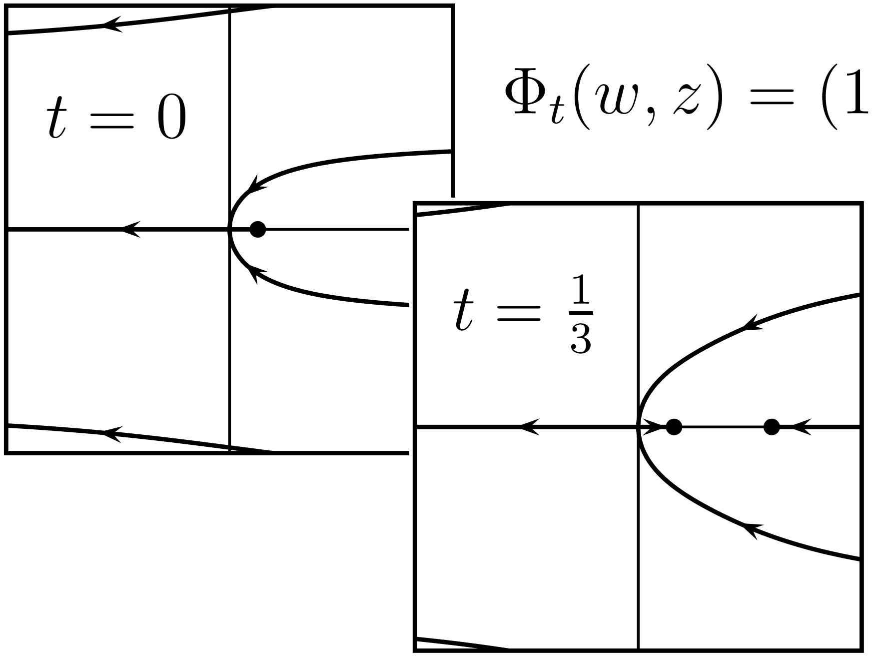

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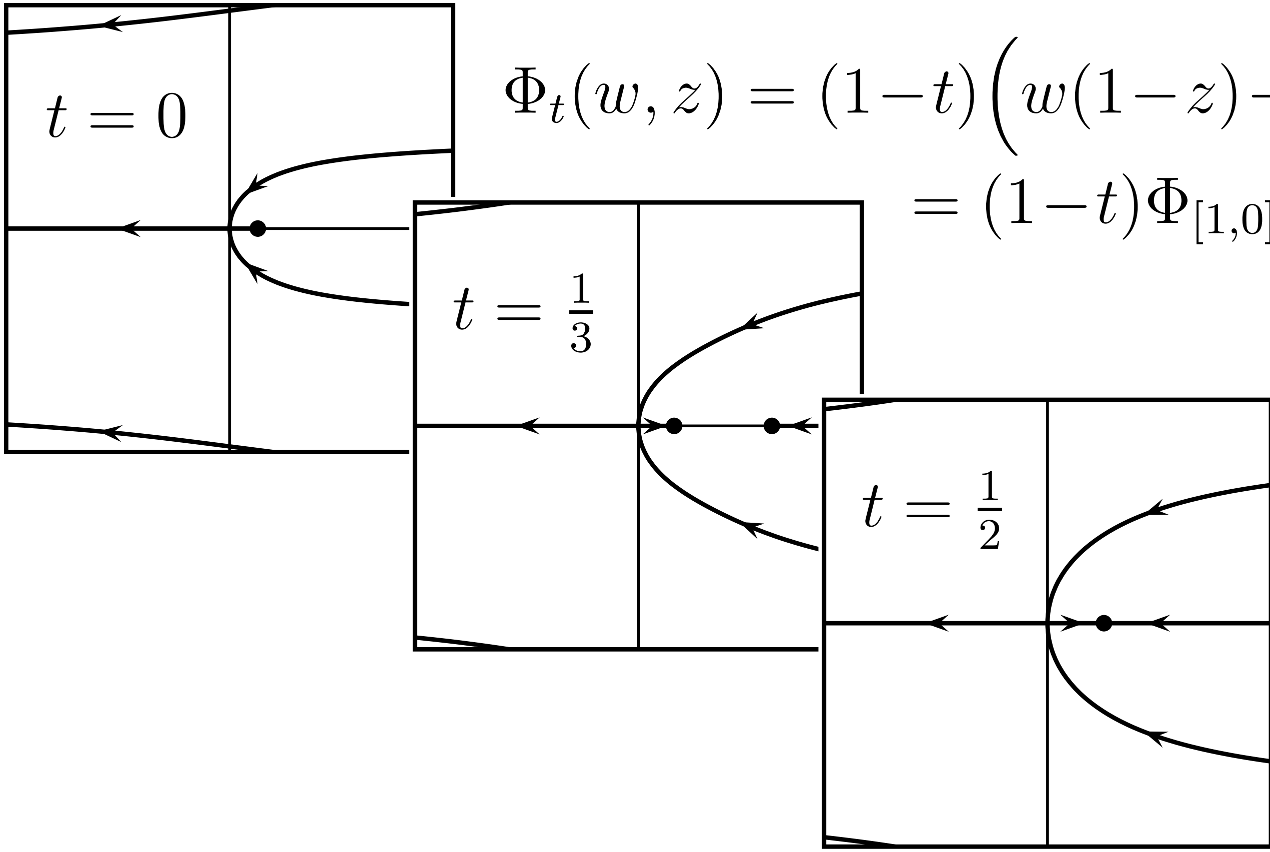

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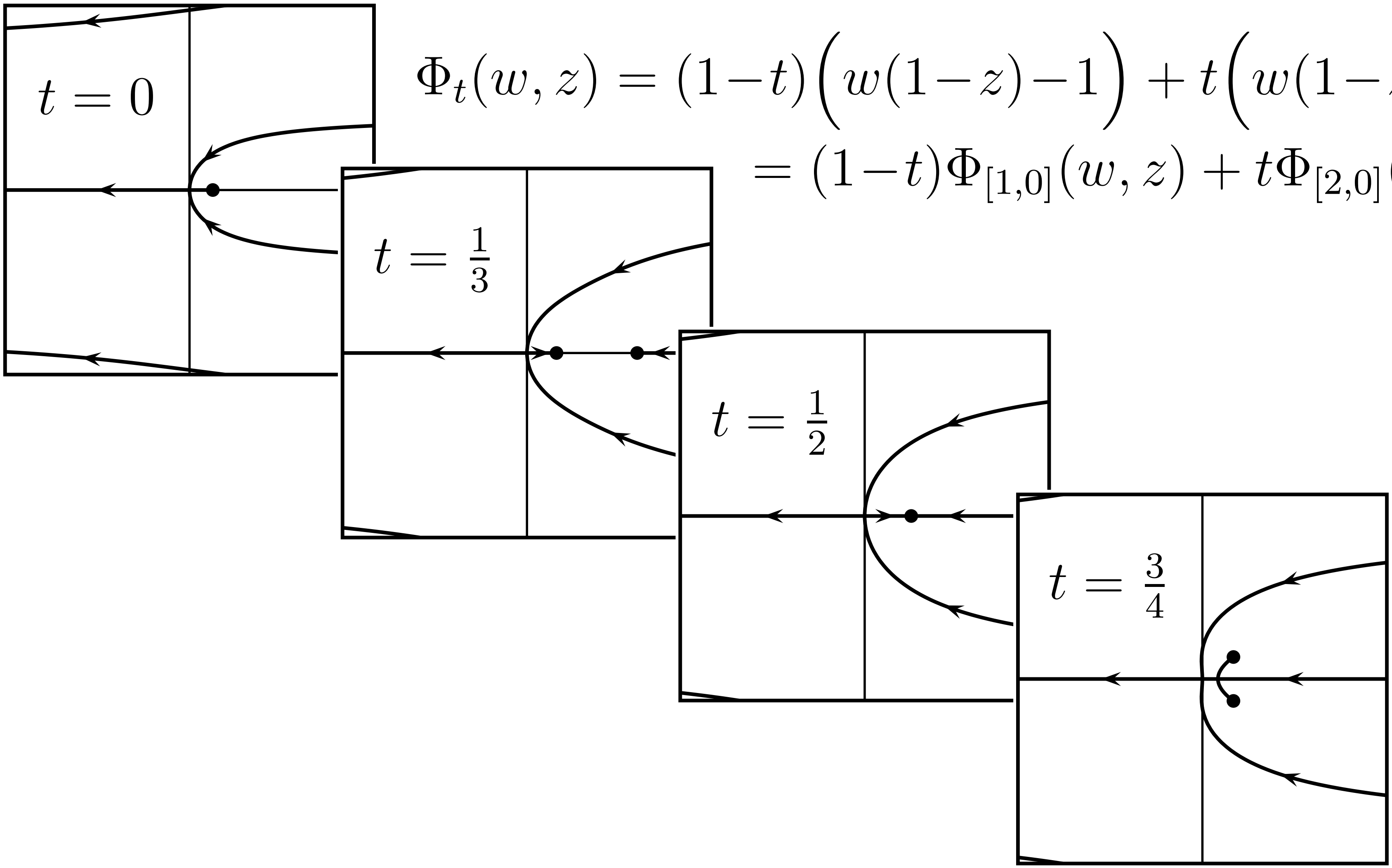

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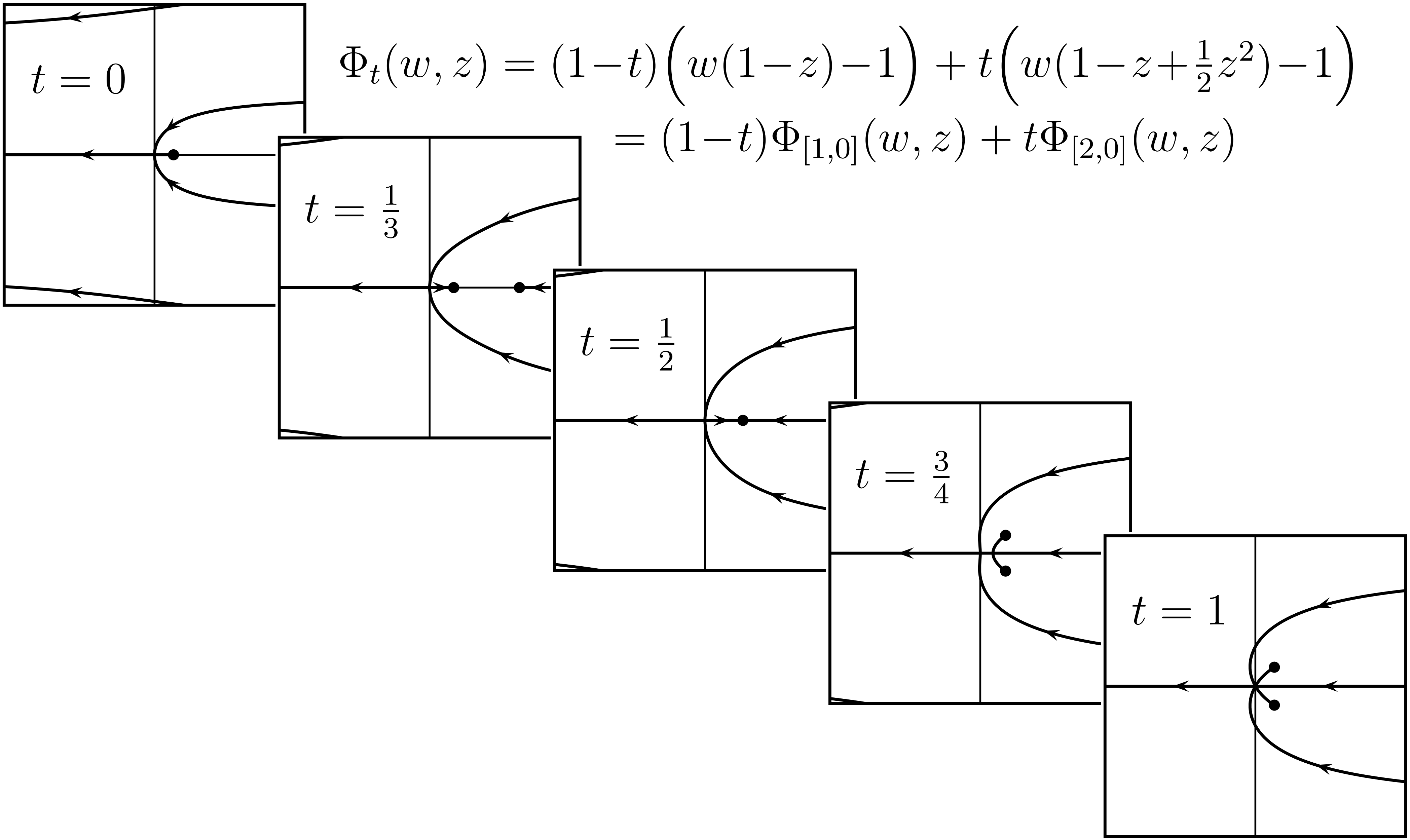

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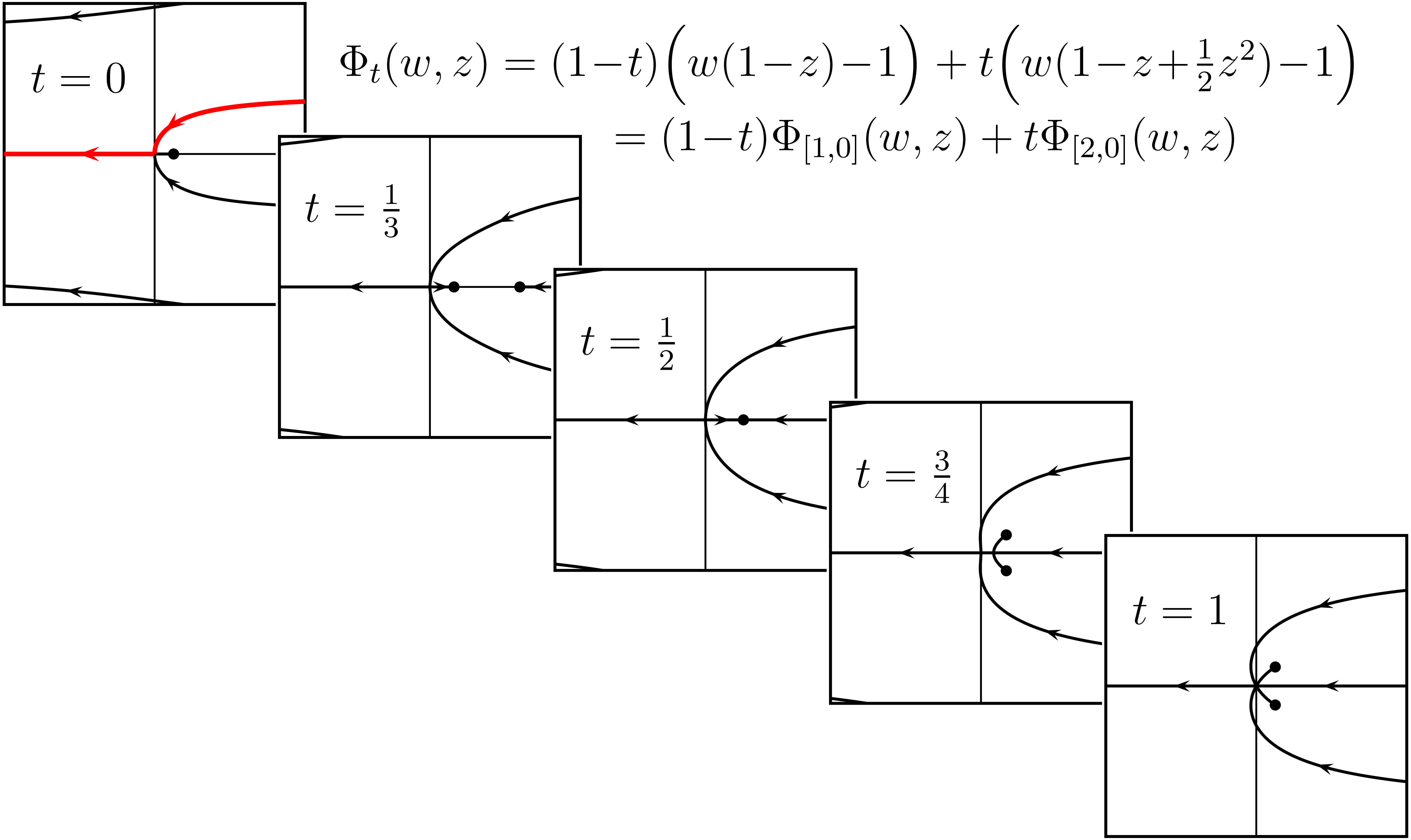

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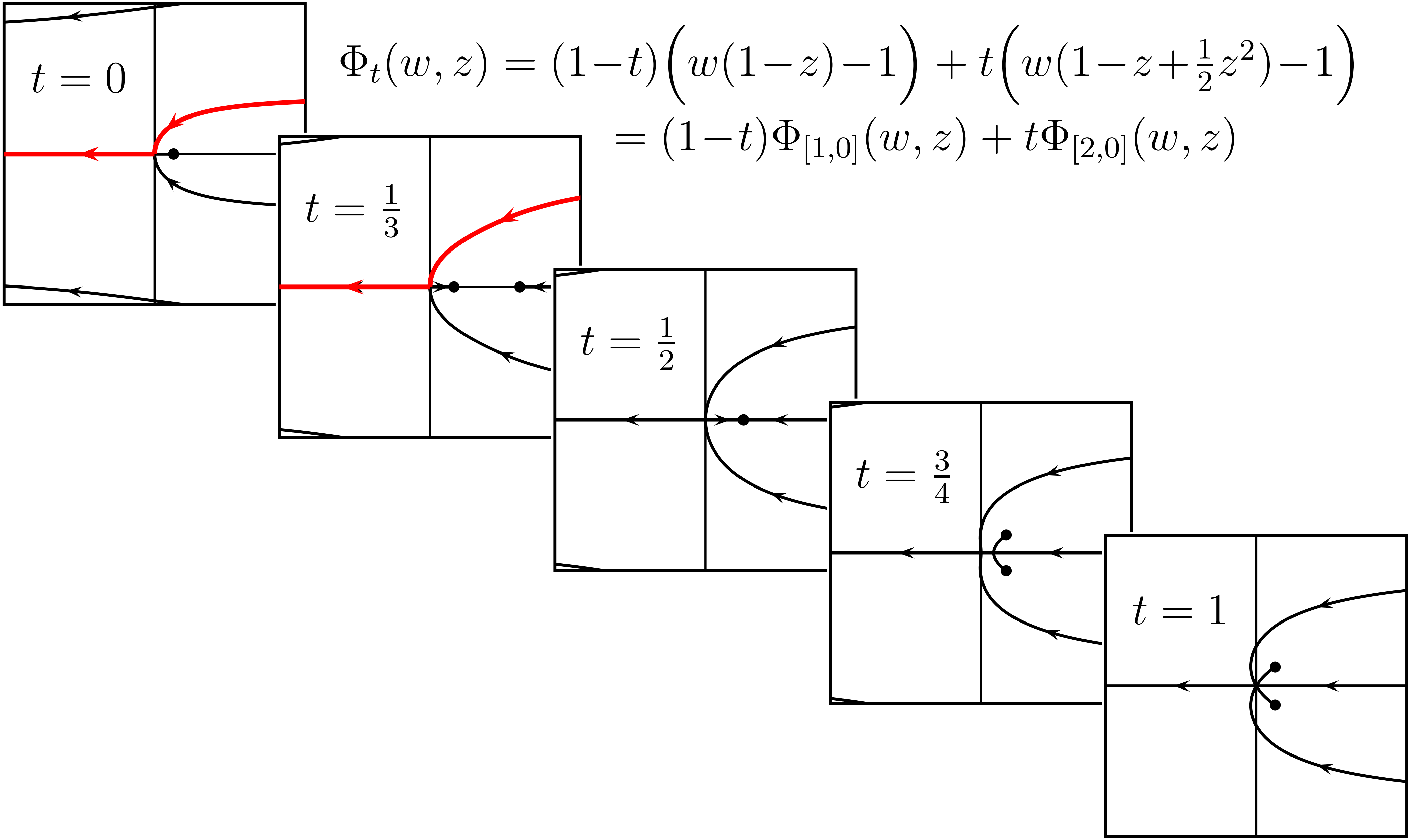

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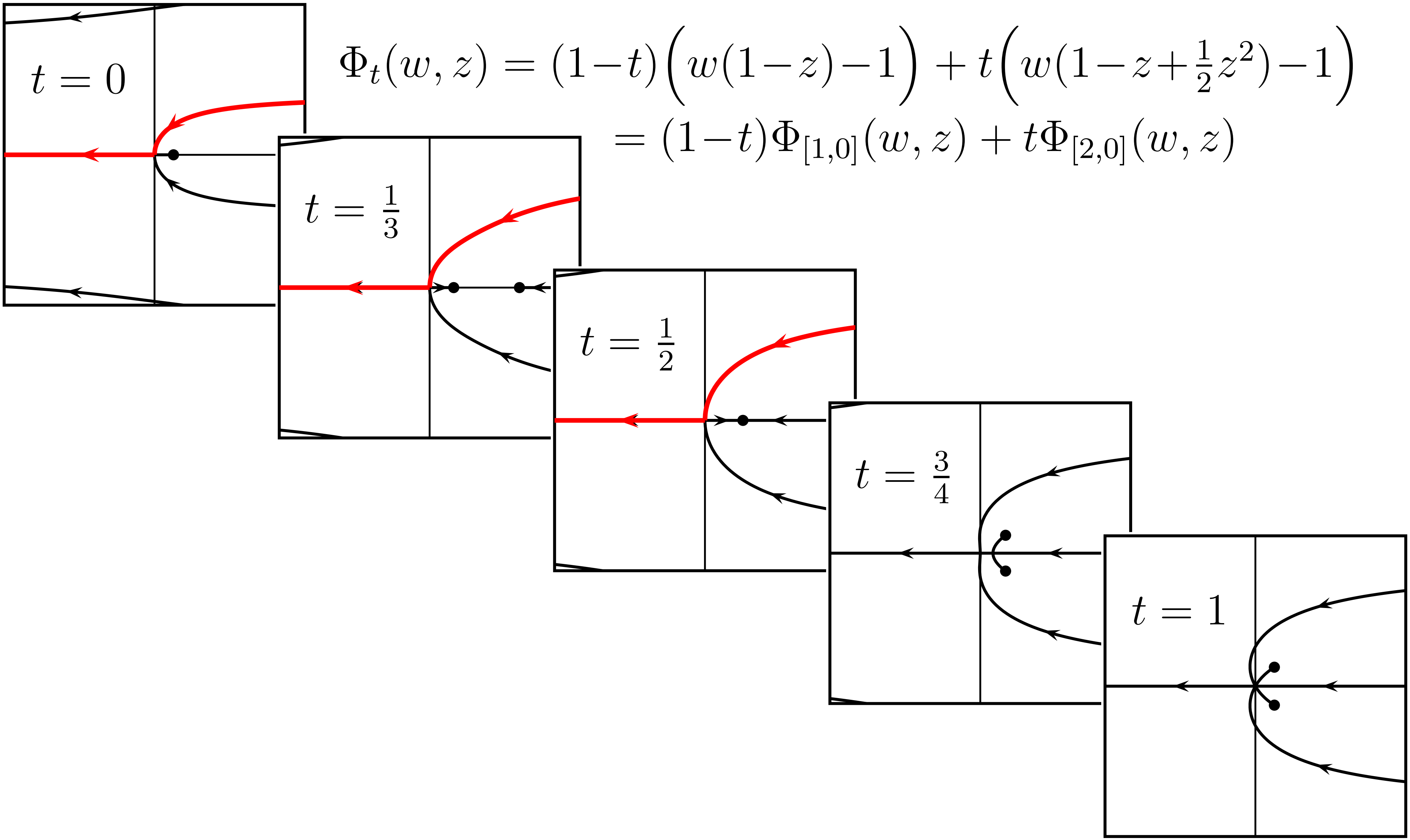

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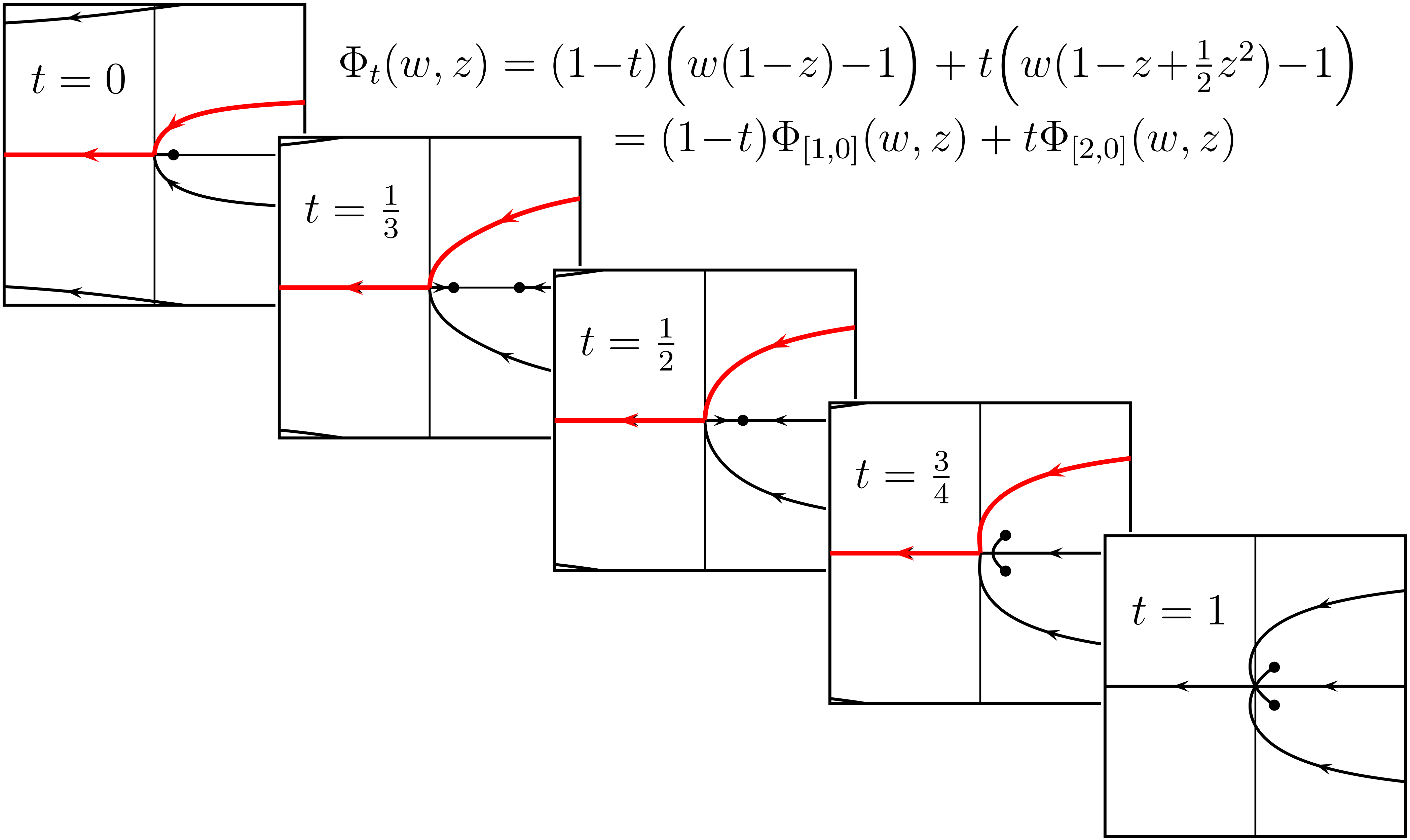

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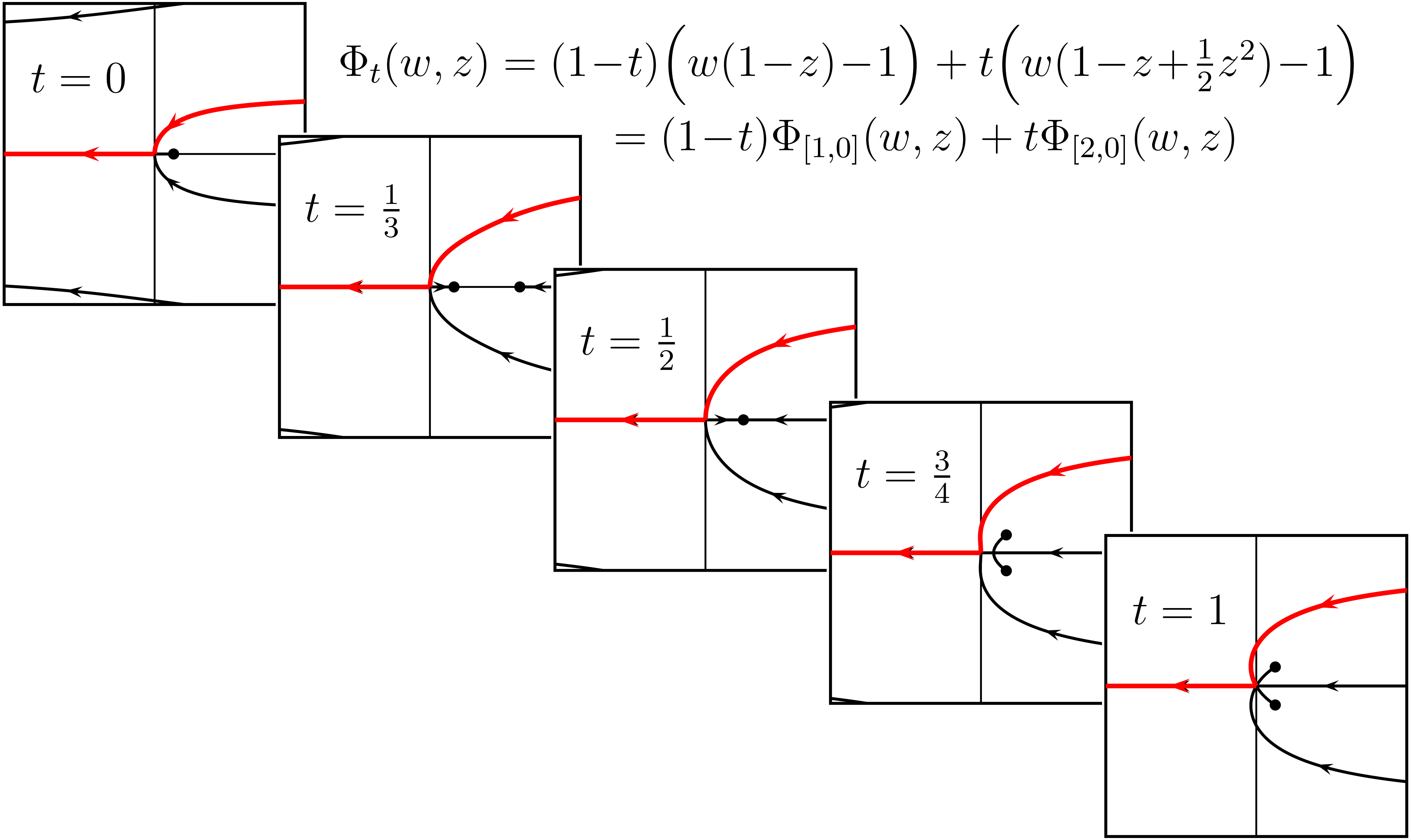

Quadratic approximations to the exponential

Given $[n_0, n_1, n_2]$, there exist polynomials $P_i, i = 0, 1, 2$, of degrees n_0, n_1, n_2 respectively, such that

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For generalized Padé approximations to the exponential function, the necessary condition for A-stability of linear Padé approximations also holds in the general case:

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- This leaves $2n_0 - p \leq 2$ as the remaining challenge
- I will concentrate on the quadratic case

We are only interested in "genuine" order. For example, consider the approximations

$$\Phi_{[2,0,2]}(w,z) = w^2\left(1-\tfrac{5}{8}z+\tfrac{1}{8}z^2\right)-2w+\left(1+\tfrac{5}{8}z+\tfrac{1}{8}z^2\right), \quad (1)$$

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To make sure that we never deal with such irrelevancies, we will consider only cases for which $n_0 > n_1 + n_2$.

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This will be illustrated by a single example:

$$\Phi_t = (1-t)\Phi_{[2,1,-1]} + t\Phi_{[2,1,0]},$$

where

$$\Phi_{[2,1,-1]}(w,z) = w^2(1 - \tfrac{2}{3}z + \tfrac{1}{6}z^2) - w(1 + \tfrac{1}{3}z),$$

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$$\left(\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\right)\Phi_t(w, z).$$

This has a factor z^3, corresponding to the order 3 stagnation point at 0, and the additional factor

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with zero at $z = -4$, when $t = 0$, and at $z = 0$, when $t = 1$.

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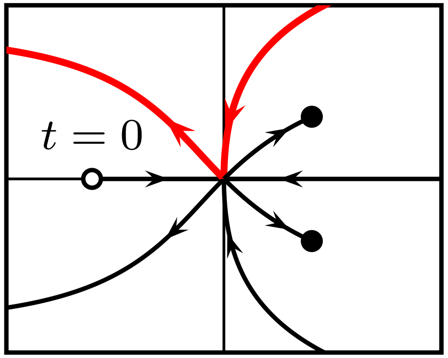

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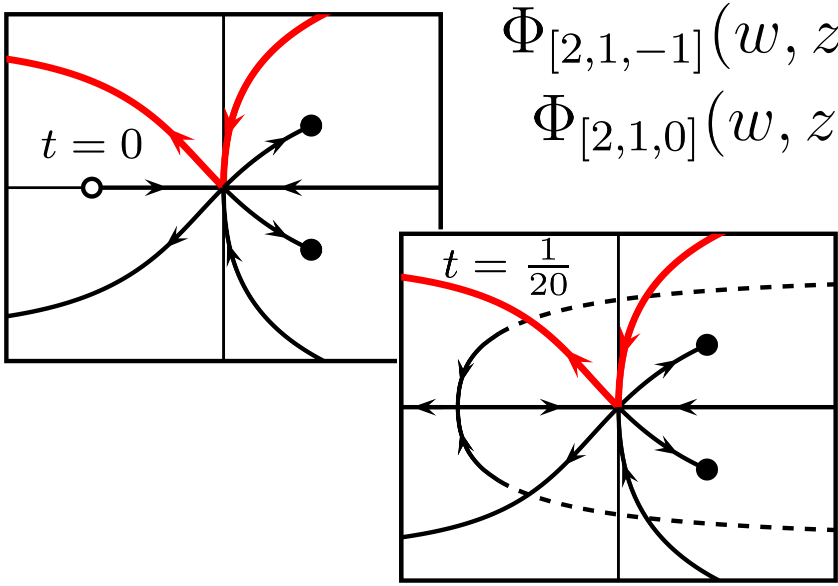

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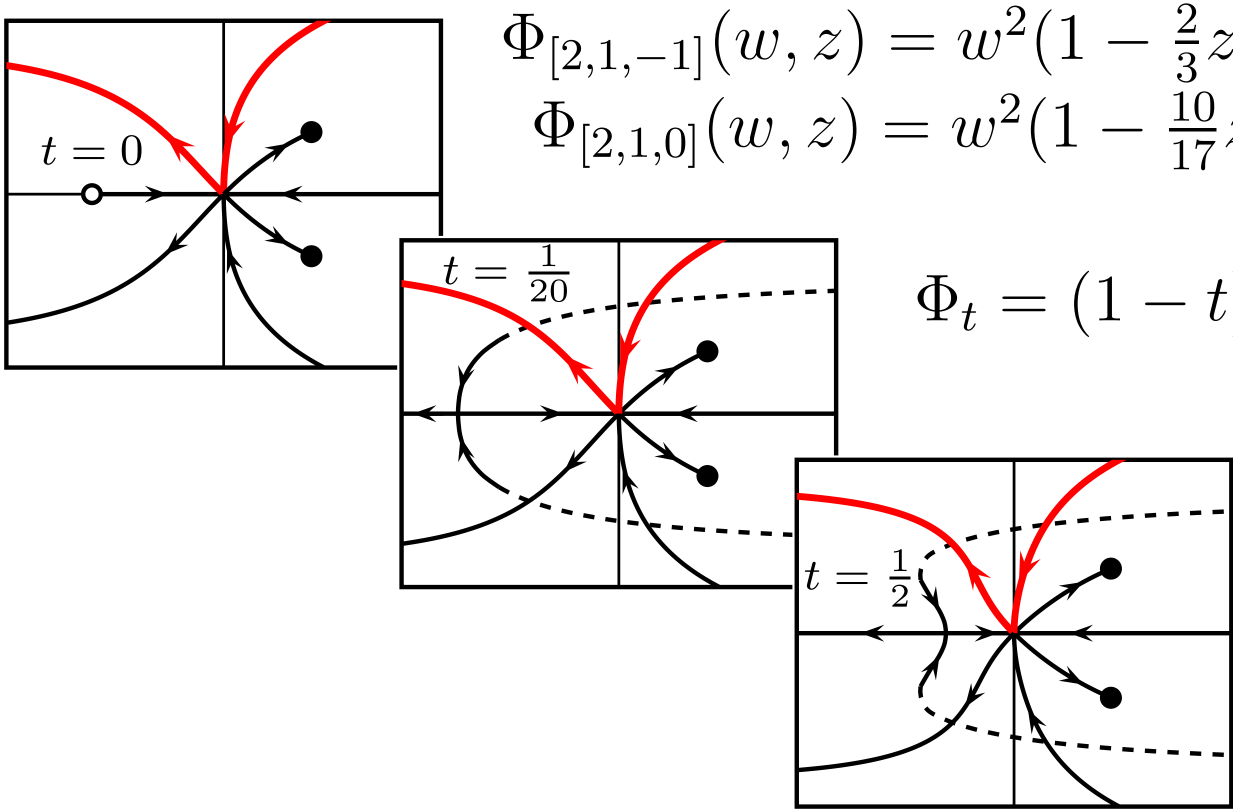

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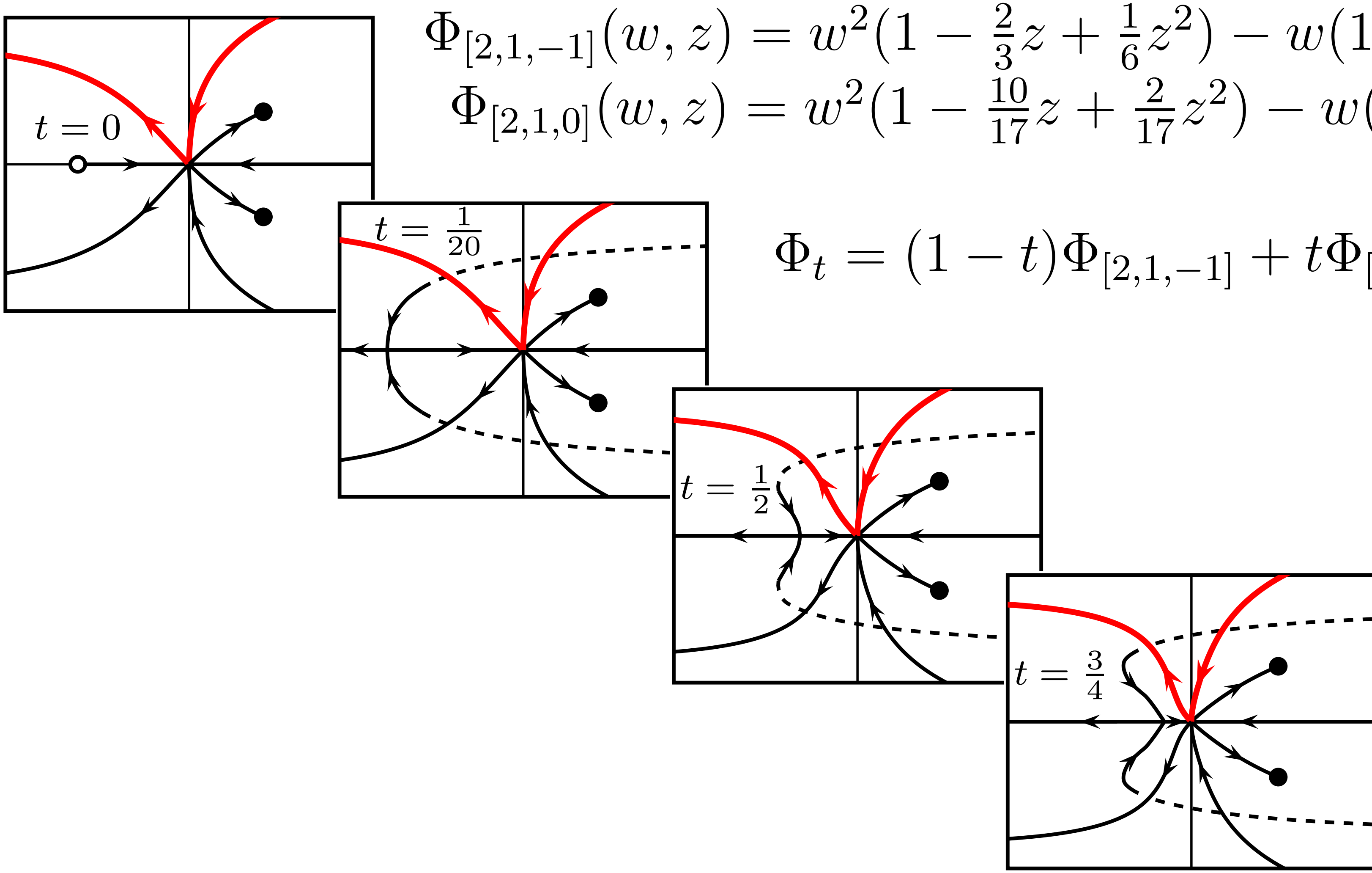

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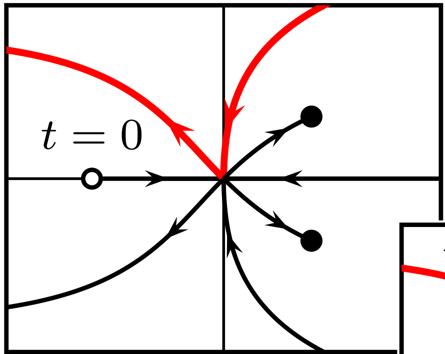

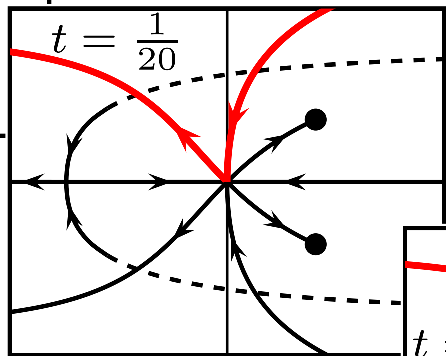

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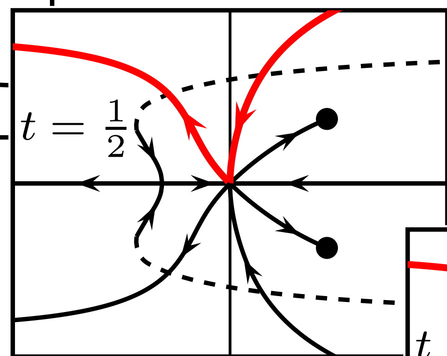

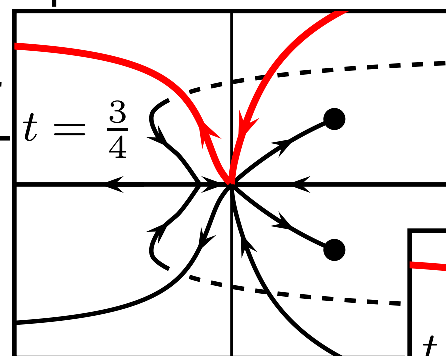

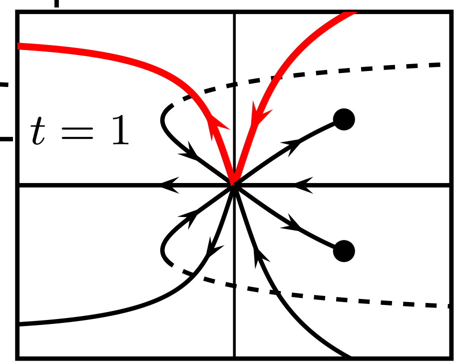

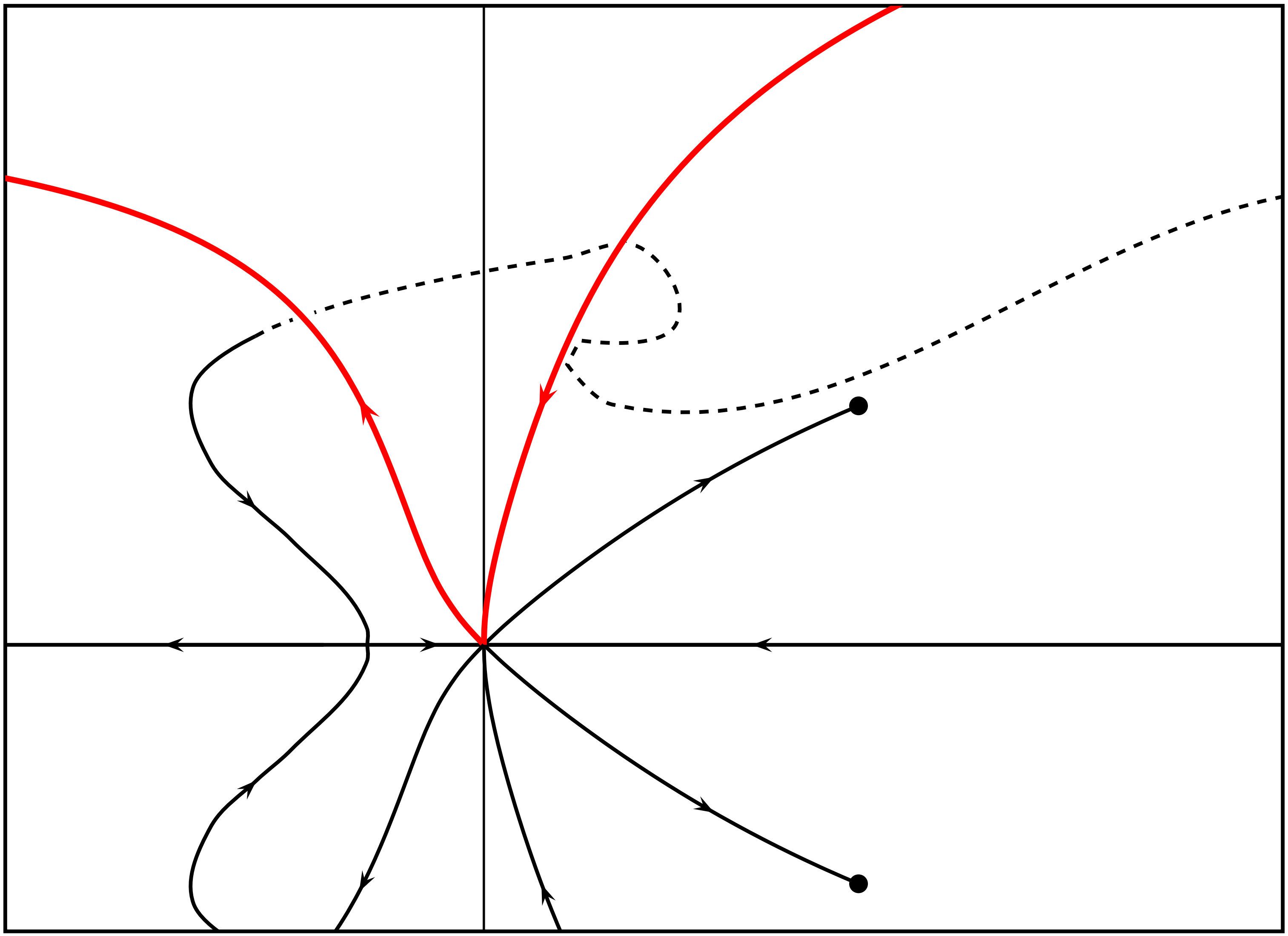

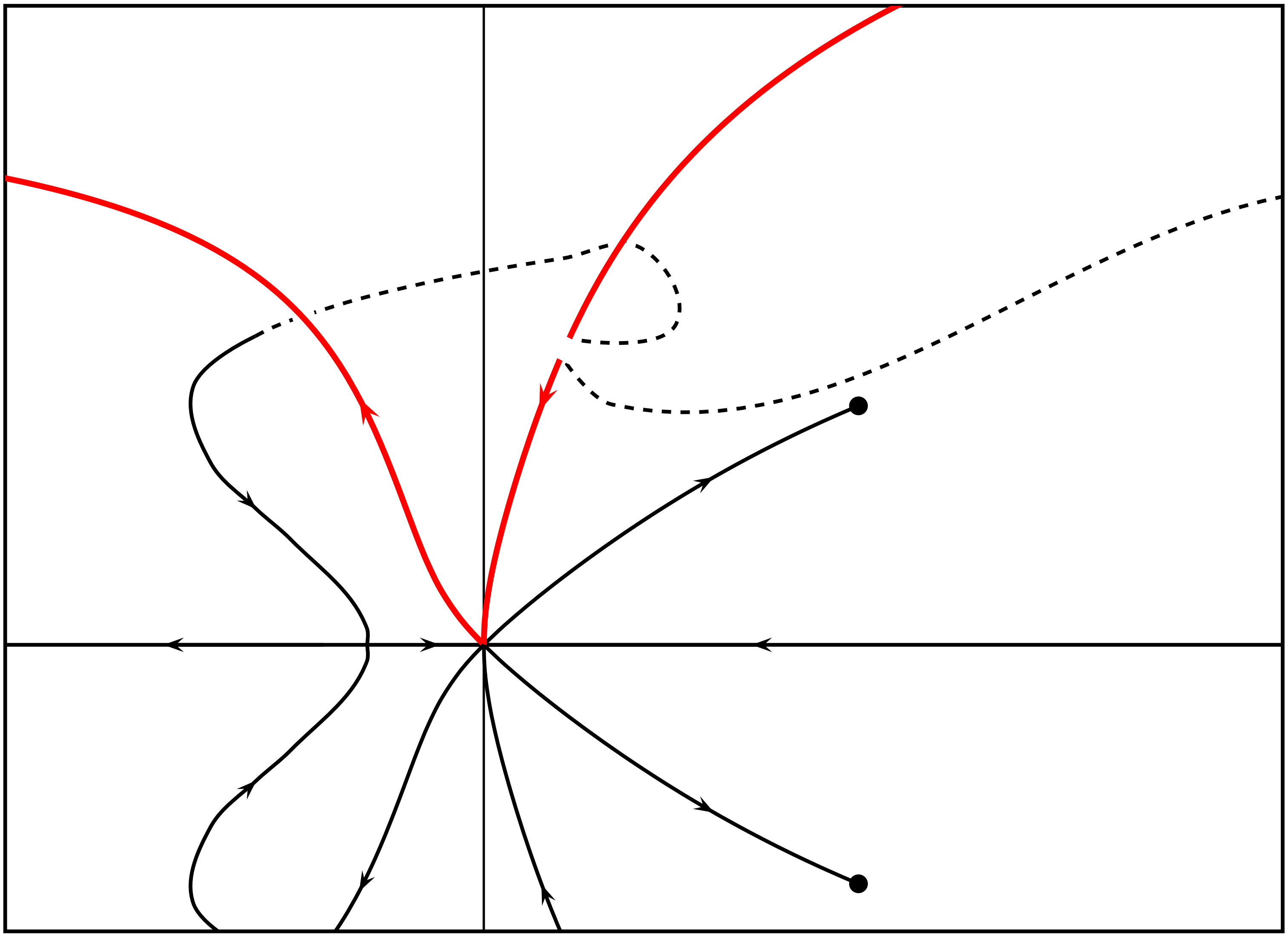

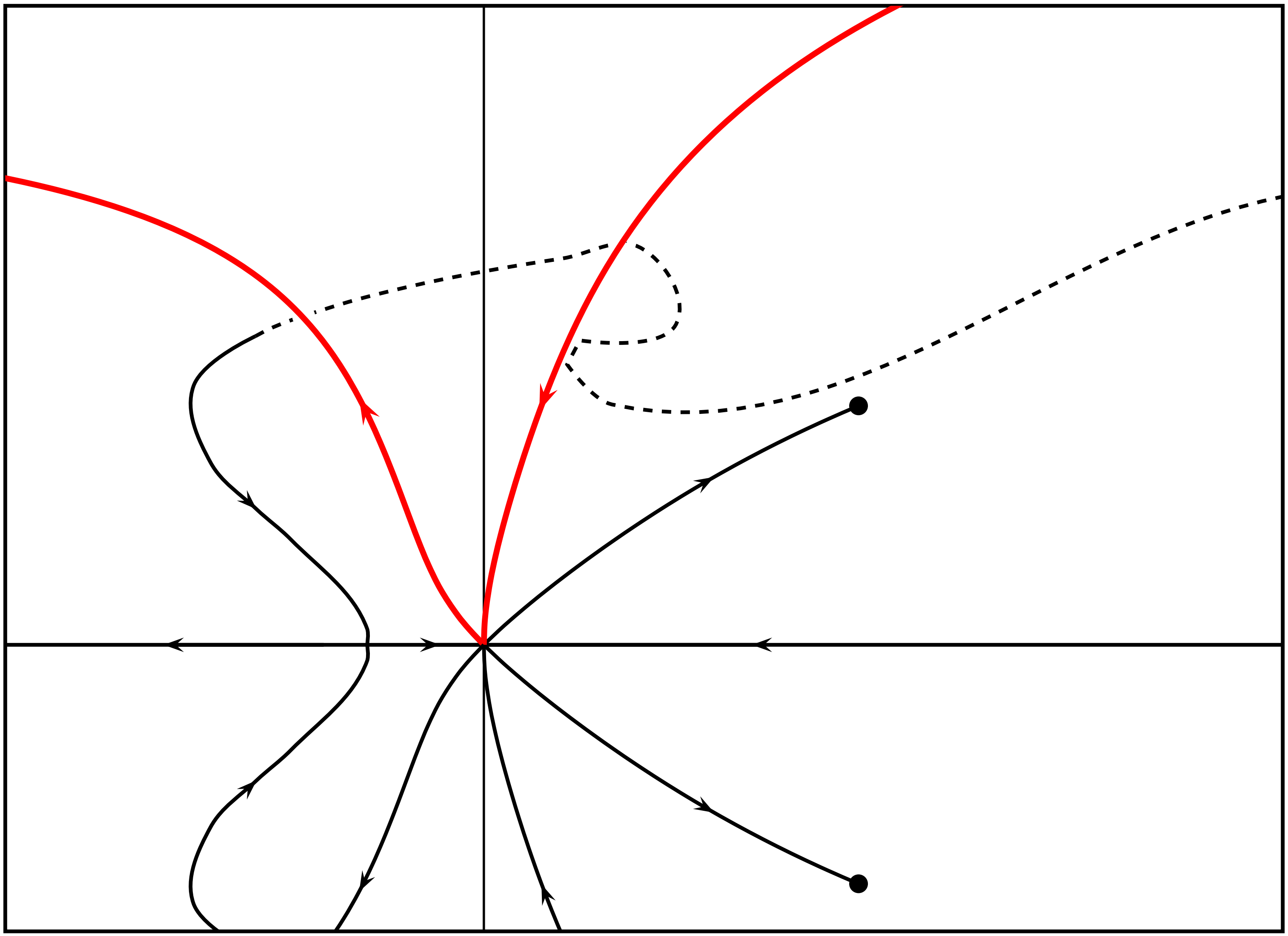

Many thanks