Thirty years of G-stability

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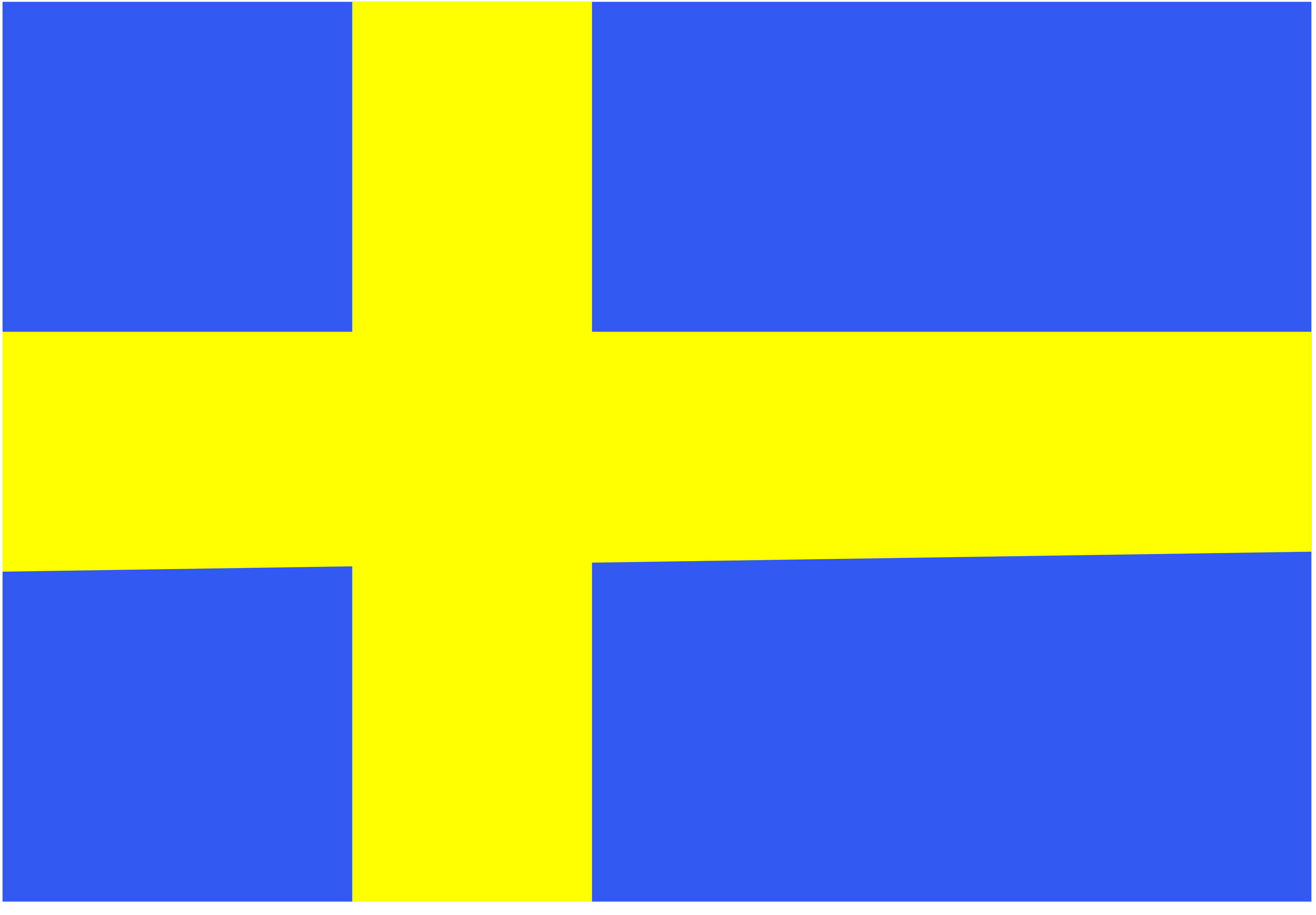
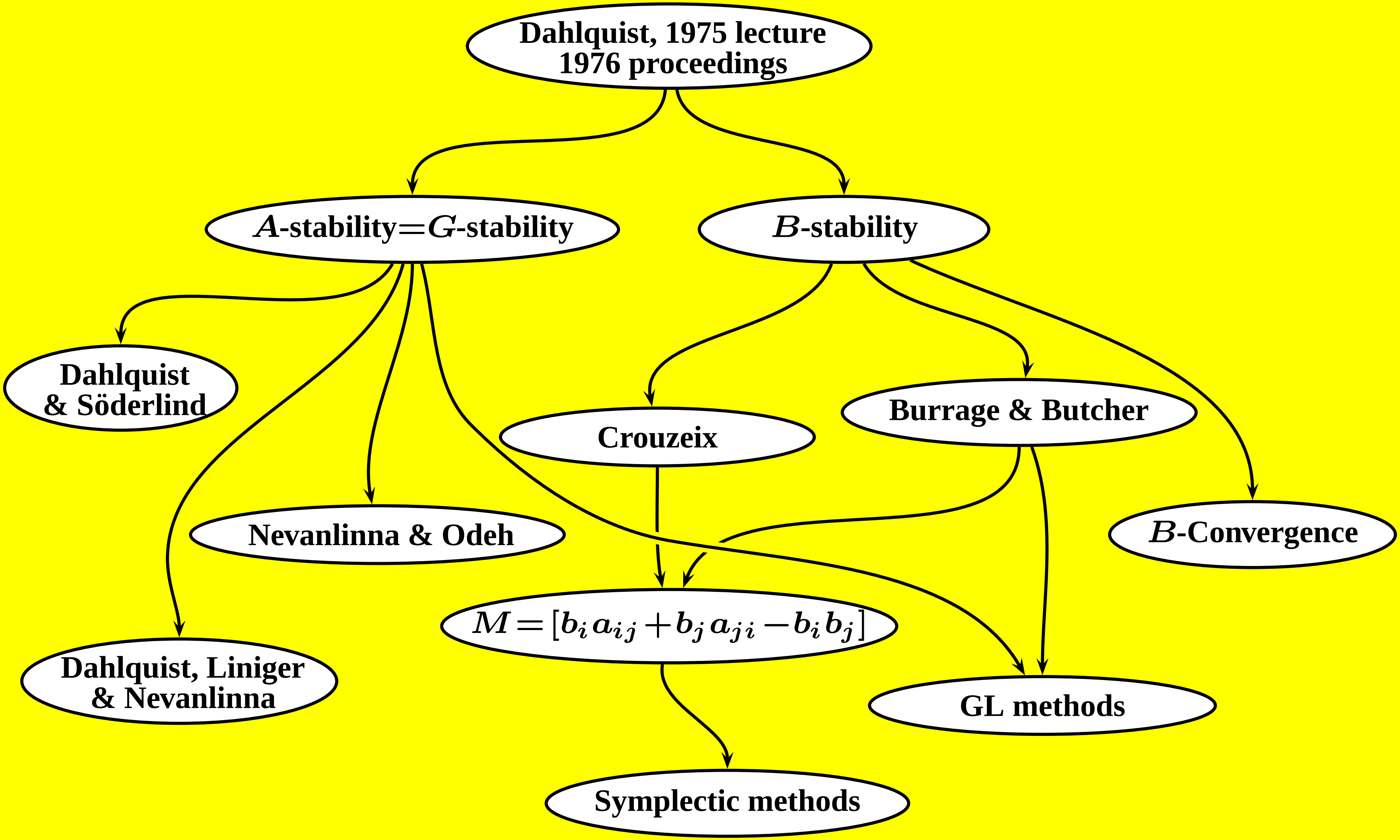
In memory of Germund Dahlquist, friend, mentor and inspiration



G-stability and consequences



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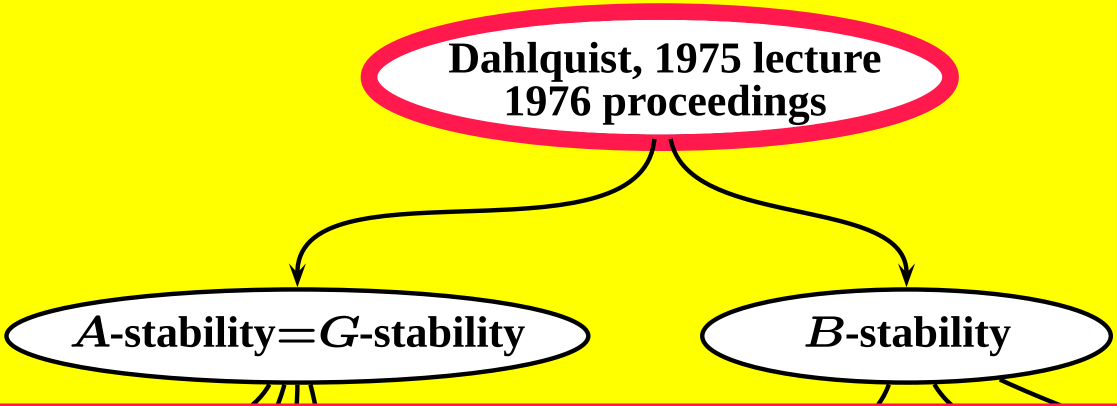
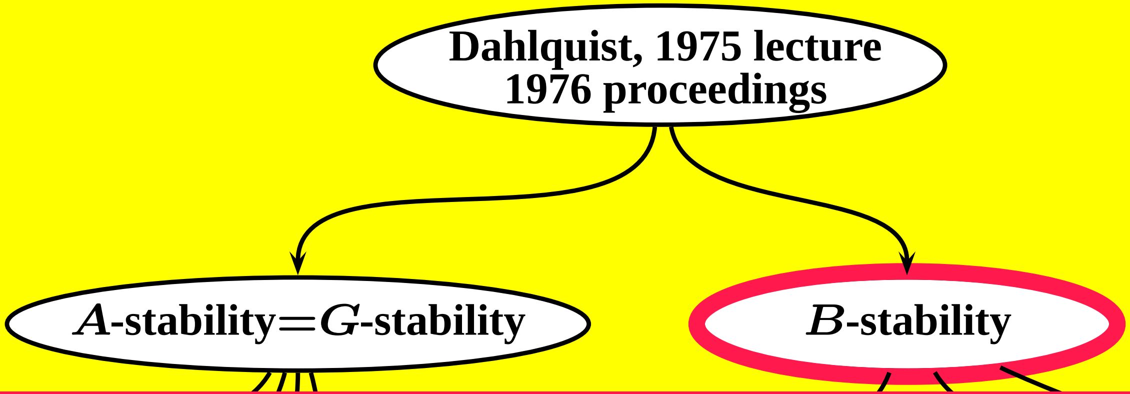


G. Dahlquist, *Error analysis of a class of methods for stiff nonlinear initial value problems*, Numerical Analysis, Dundee, Lecture Notes in Mathematics **506**, (1976) 60–74.

- Introduced "One-leg" counterpart to linear multistep method.
- Considered the test problem $dy/dx = f(x, y)$, where $\langle y - z, f(x, y) - f(x, z)\rangle \leq 0$.
- Sought G such that $\|Y_{n+1} - Z_{n+1}\|_G \leq \|Y_n - Z_n\|_G$, where $Y_n = \big[y_n, y_{n+1}, \ldots, y_{n+k-1}\big]$.

G-stability and consequences



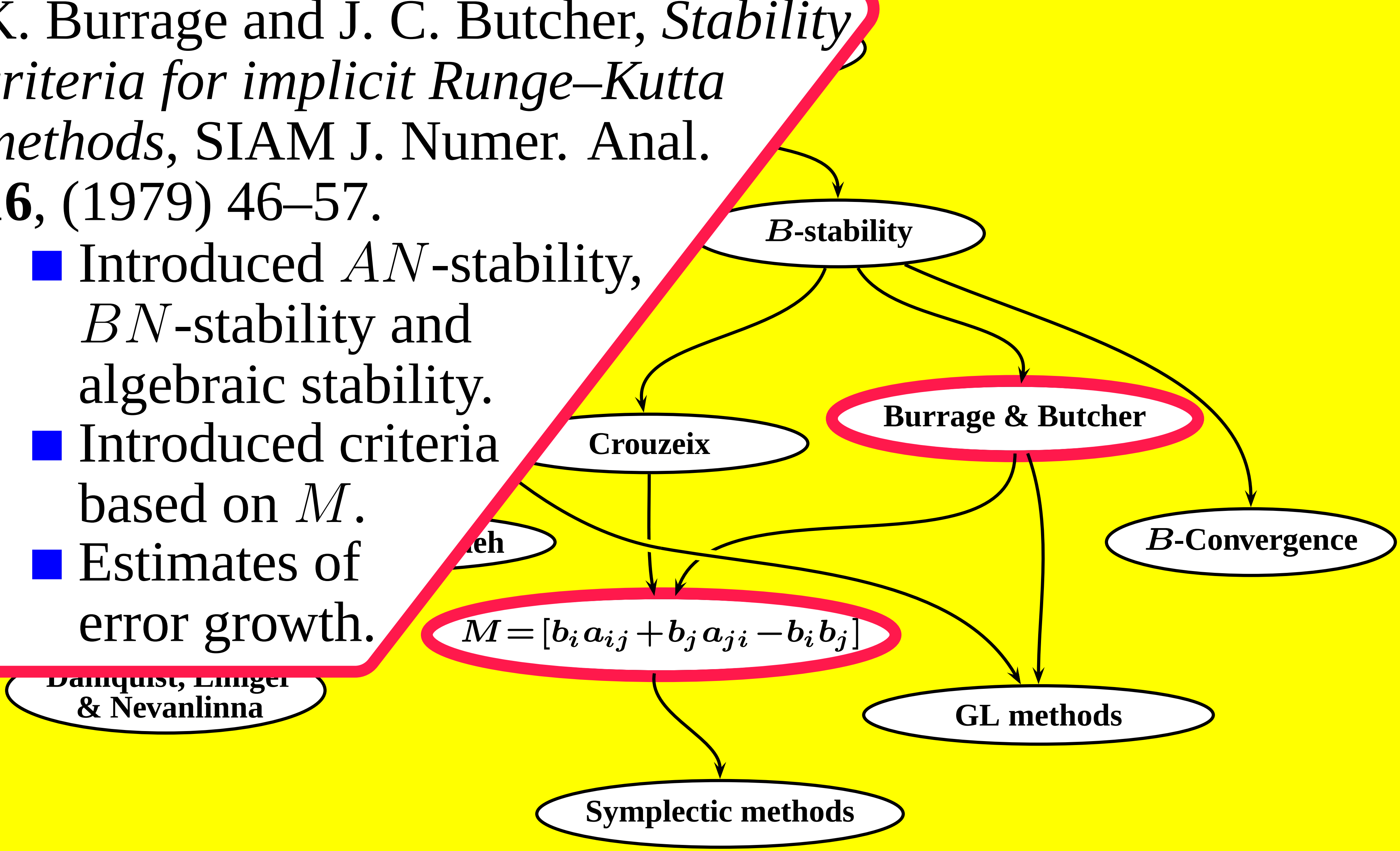
J. C. Butcher, *A stability property of implicit Runge–Kutta methods*, BIT **15**, (1975) 358–361.

- Applied same non-linear test problem to Runge–Kutta methods.
- Considered only methods (A, b^T, c) with A non-singular.
- Complicated criteria found for B-stability.
- Various standard methods of orders $2s$, $2s-1$ and $2s-2$ shown to be B-stable.

G-stability and consequences

K. Burrage and J. C. Butcher, *Stability criteria for implicit Runge–Kutta methods*, SIAM J. Numer. Anal. **16**, (1979) 46–57.

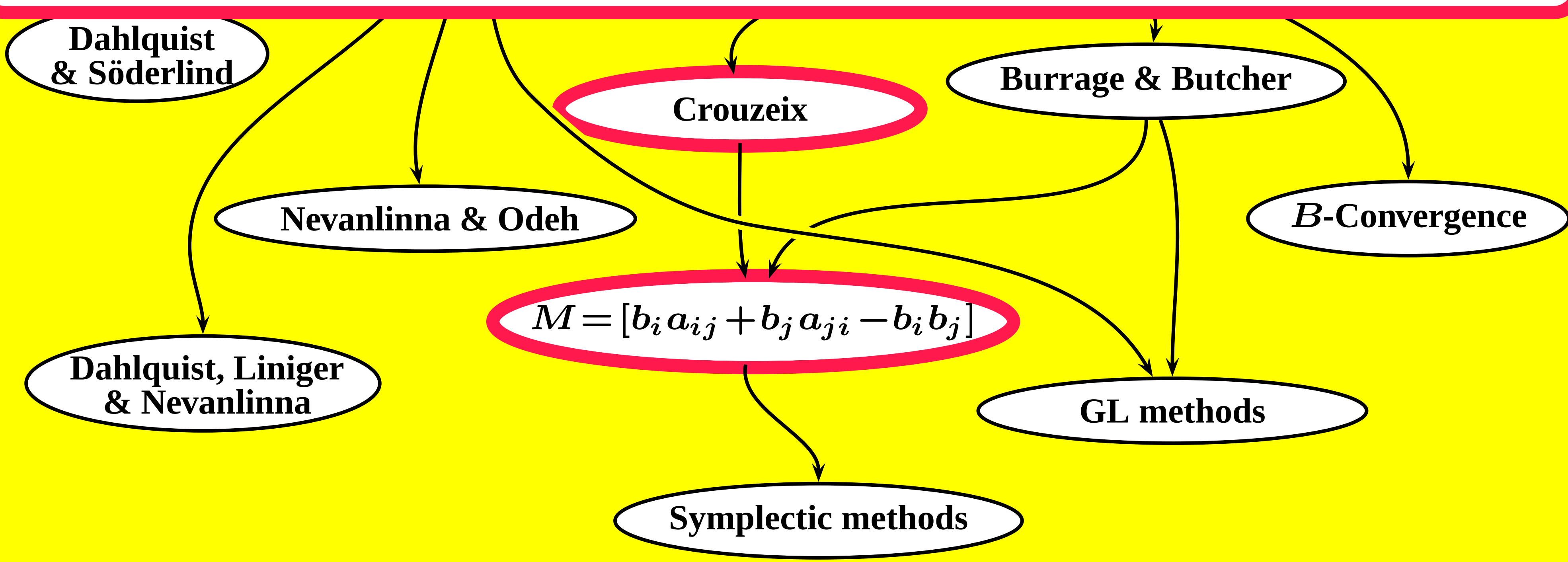
- Introduced AN-stability, BN-stability and algebraic stability.
- Introduced criteria based on M.
- Estimates of error growth.



G-stability and consequences

M. Crouzeix, *Sur la B-stabilité des méthodes de Runge–Kutta*, Numer. Math. **32**, (1979) 75–82.

- Introduces criterion for B-stability based on matrix M.



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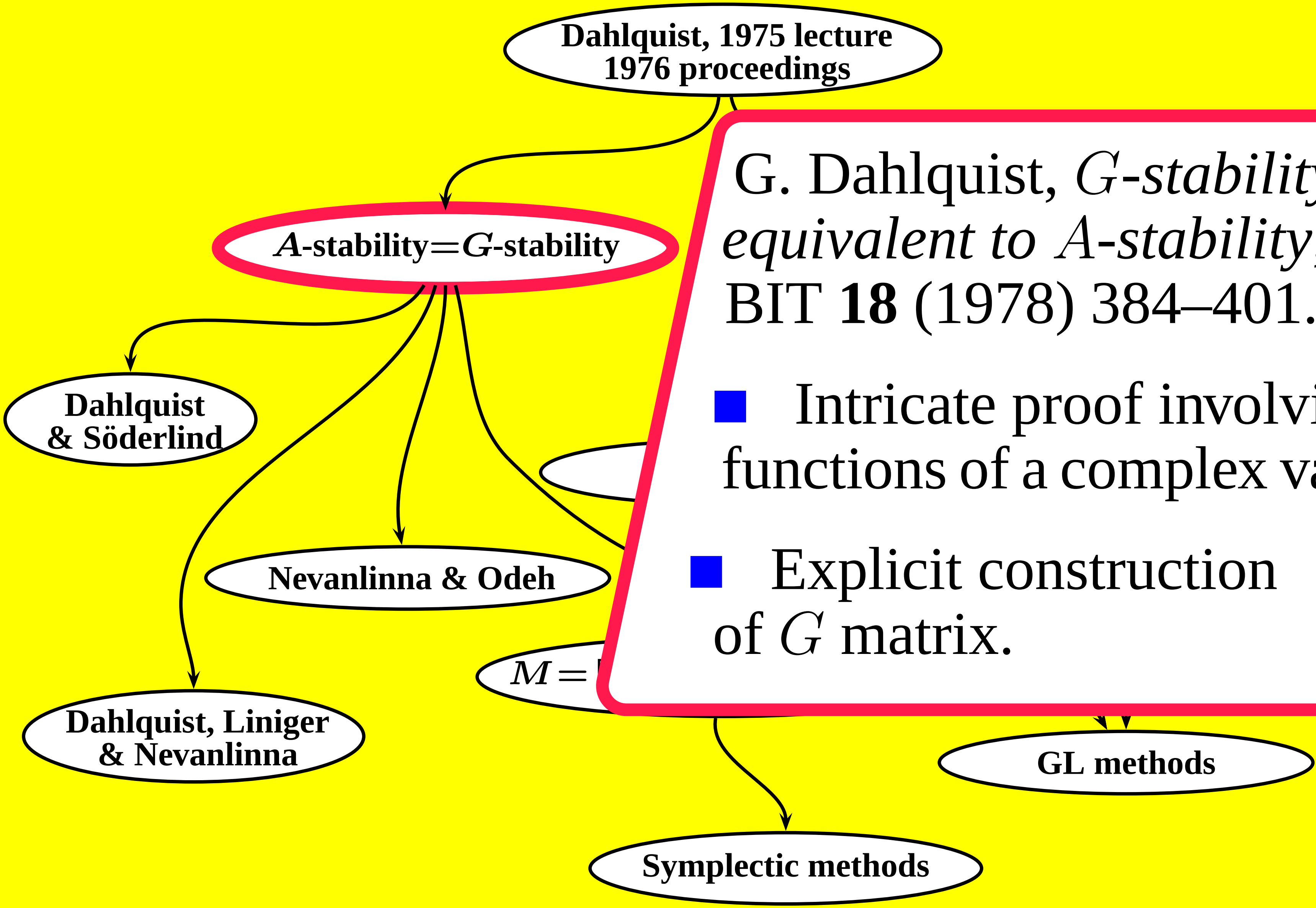
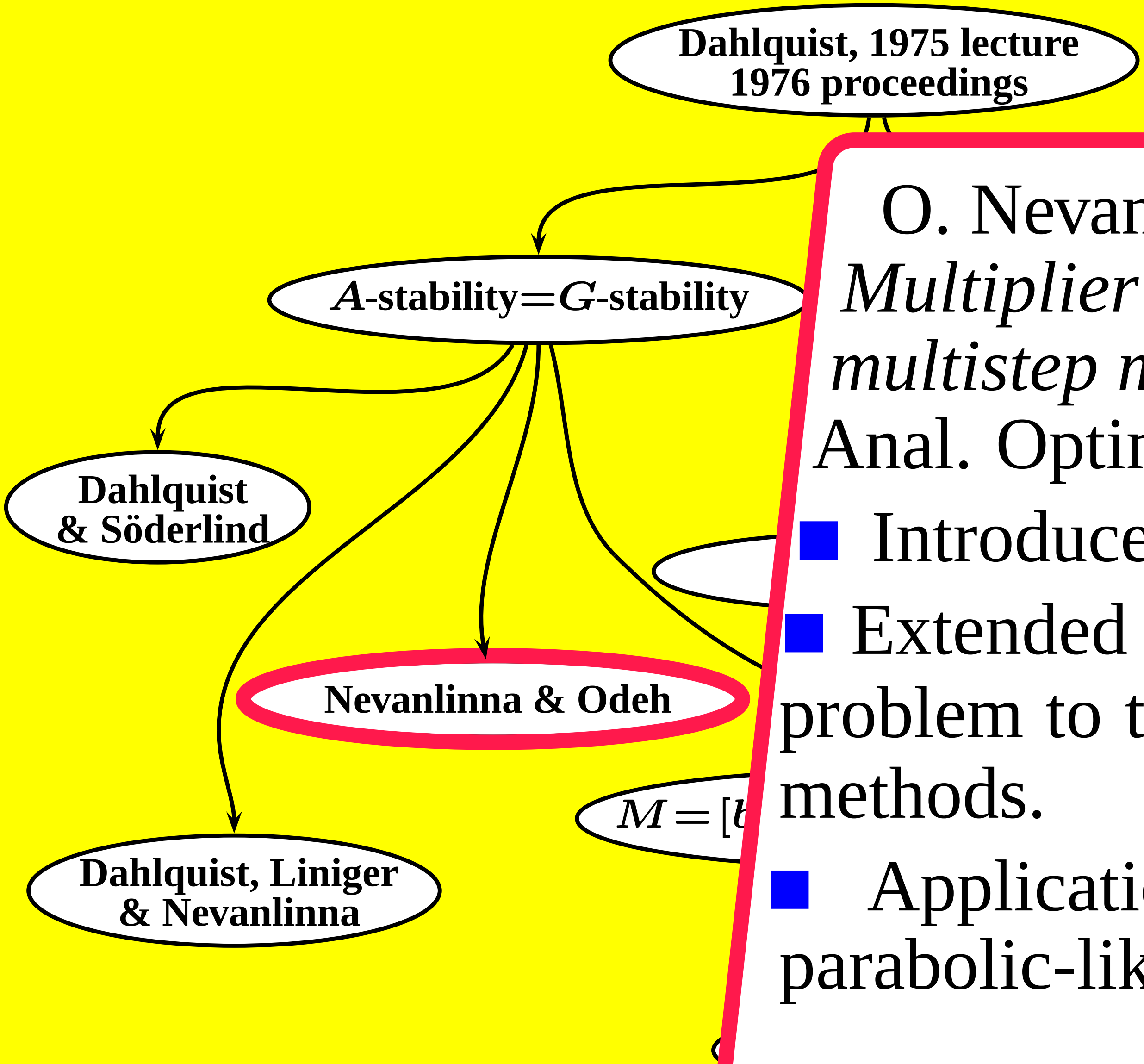


G. Dahlquist, *G-stability is equivalent to A-stability*, BIT **18** (1978) 384–401.

- Intricate proof involving functions of a complex variable.
- Explicit construction of G matrix.

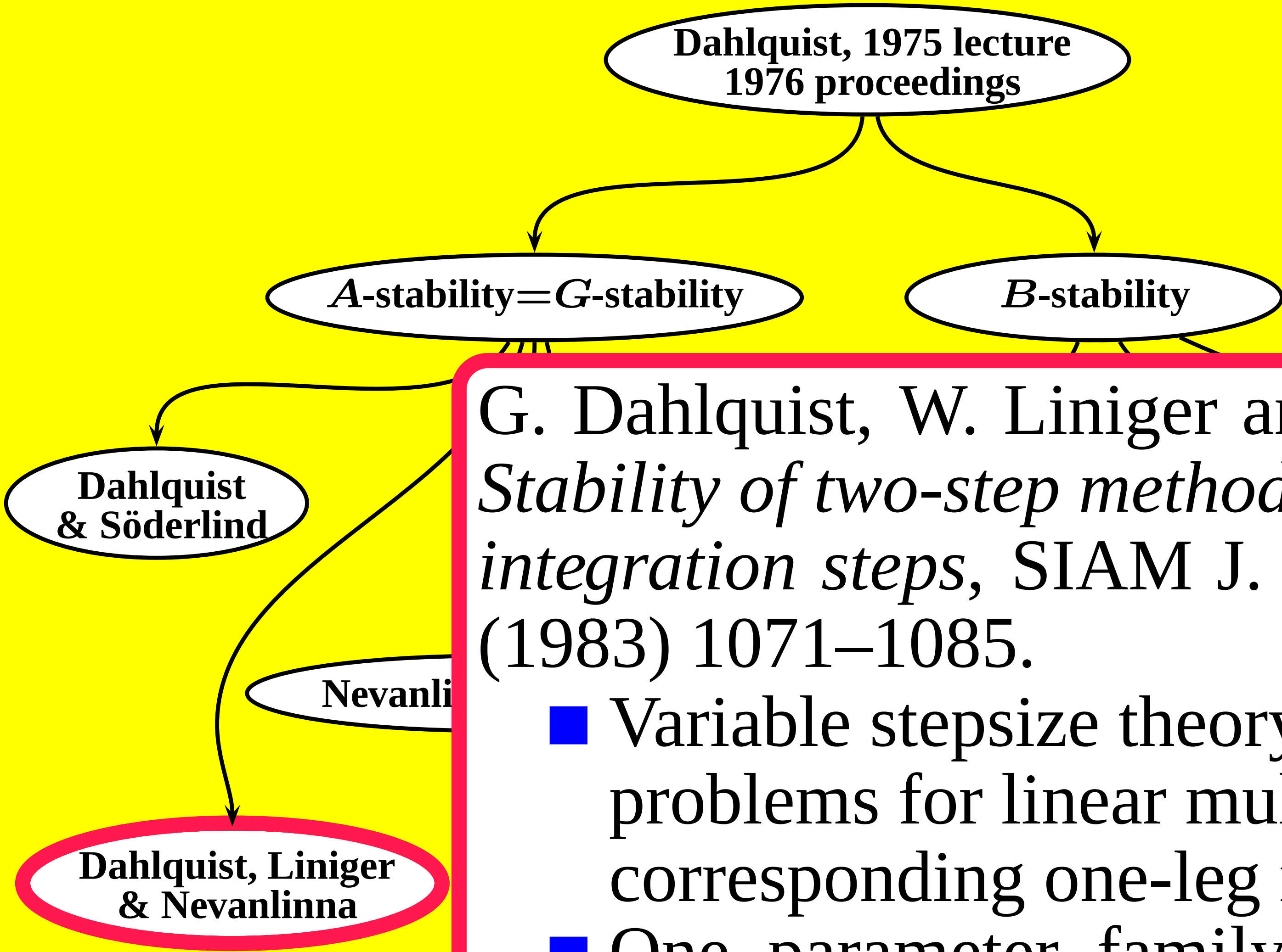
G-stability and consequences



O. Nevanlinna and F. Odeh, *Multiplier techniques for linear multistep methods*, Numer. Funct. Anal. Optim. **3** (1981) 377–423.

- Introduces multiplier techniques.
- Extended Dahlquist's model problem to the study of $A(\alpha)$-stable methods.
- Applications to nonlinear parabolic-like problems.

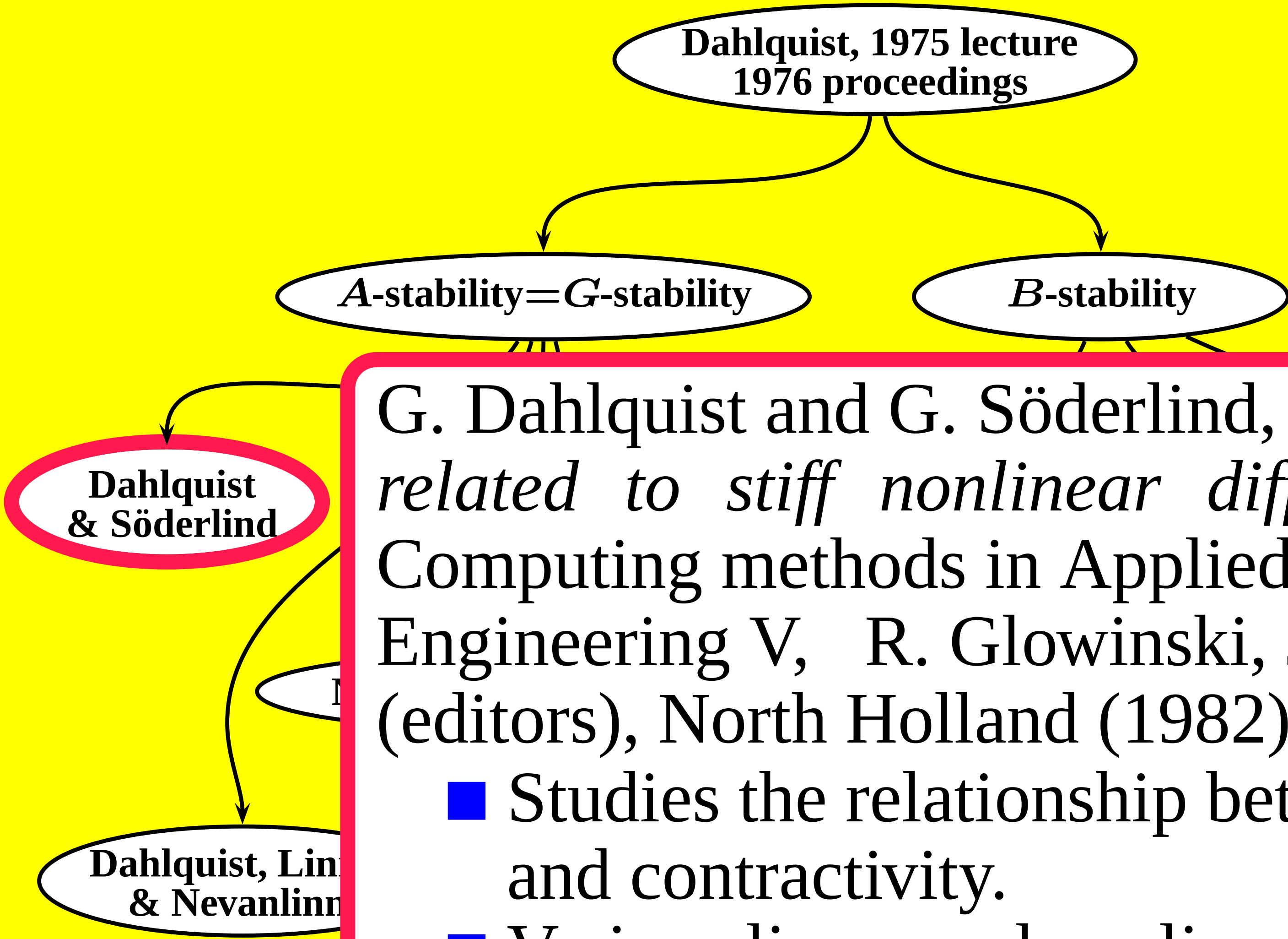
G-stability and consequences



G. Dahlquist, W. Liniger and O. Nevanlinna, *Stability of two-step methods for variable integration steps*, SIAM J. Numer. Anal. **20** (1983) 1071–1085.

- Variable stepsize theory for dissipative problems for linear multistep and corresponding one-leg methods.
- One parameter family of second order methods stable for any stepsize sequence.

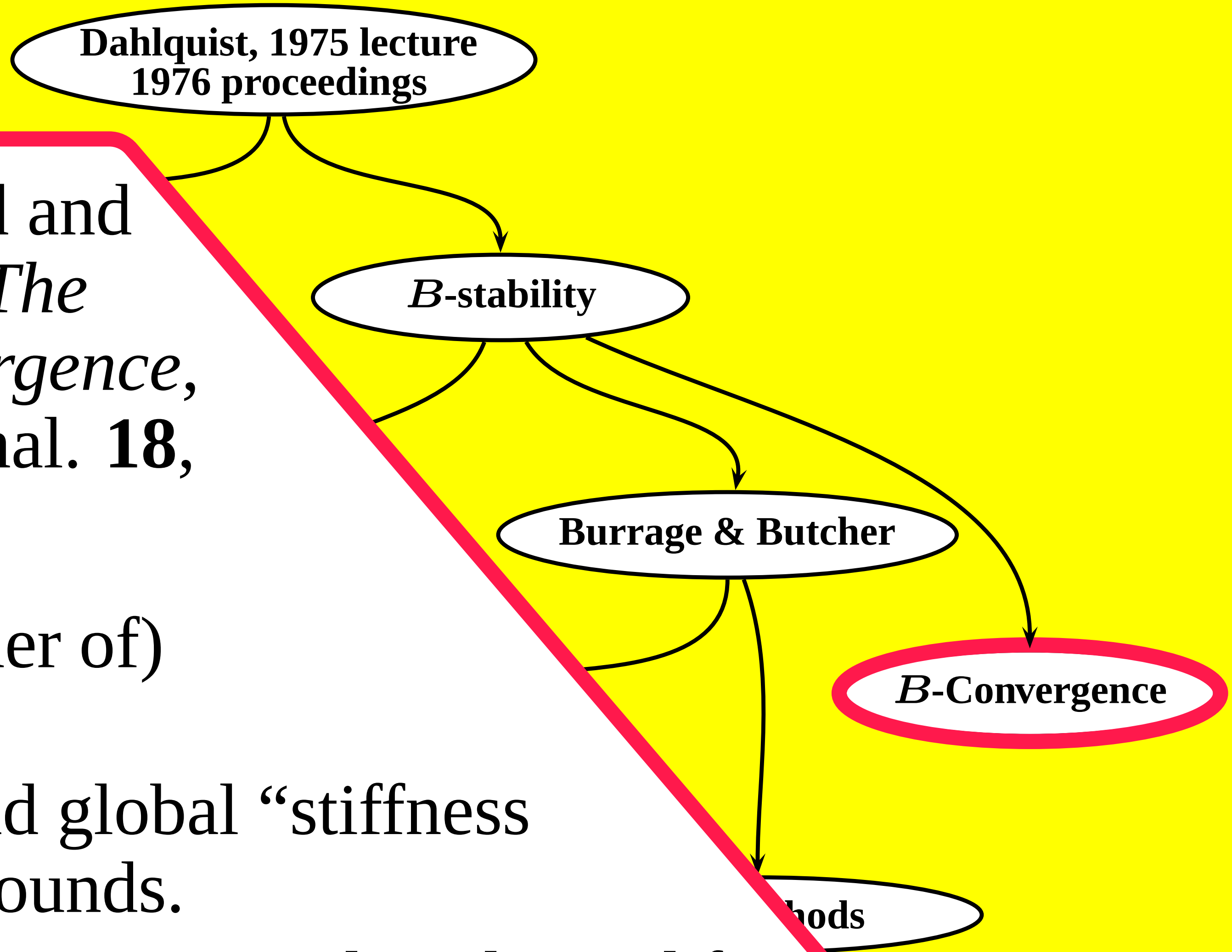
G-stability and consequences



G. Dahlquist and G. Söderlind, *Some problems related to stiff nonlinear differential systems*, Computing methods in Applied Sciences and Engineering V, R. Glowinski, J. L. Lions (editors), North Holland (1982) 57–74.

- Studies the relationship between stability and contractivity.
- Various linear and nonlinear model problems analysed.

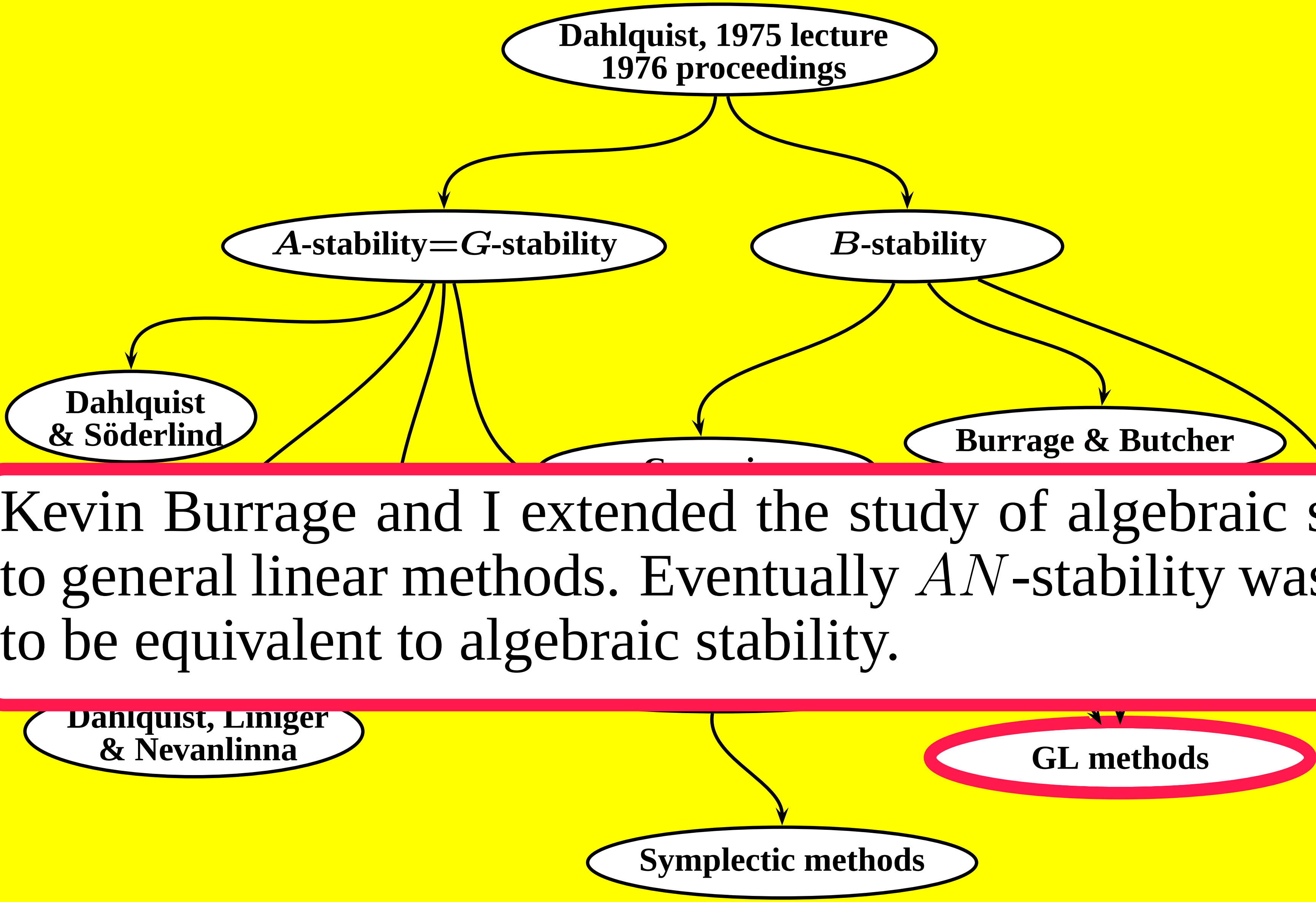
G-stability and consequences



R. Frank, J. Schneid and C. W. Ueberhuber, *The concept of B-convergence*, SIAM J. Numer. Anal. **18**, (1981) 753–780.

- Introduced (order of) B-consistency.
- Sought local and global “stiffness independent” bounds.
- Specific B-convergence orders derived for many standard method classes.

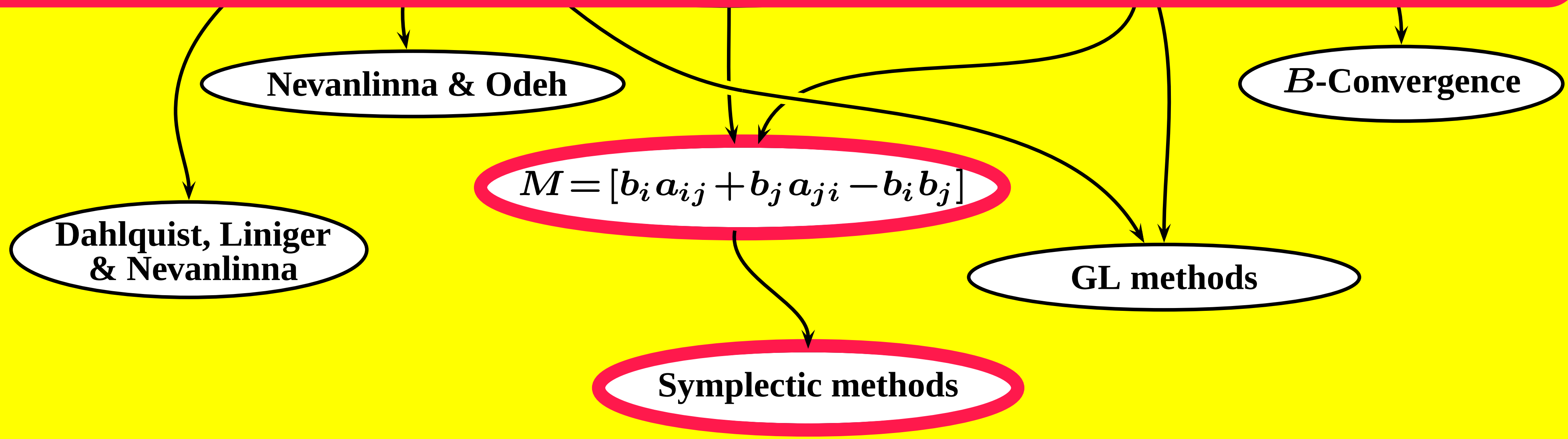
G-stability and consequences



Kevin Burrage and I extended the study of algebraic stability to general linear methods. Eventually AN-stability was shown to be equivalent to algebraic stability.

G-stability and consequences

J. M. Sanz-Serna (1988) discovered that a Runge–Kutta method is symplectic if $M = 0$.
This result was also found independently by F. Lasagni (1988) and Y. B. Suris (1989).
The following reference is to a survey paper:
J. M. Sanz-Serna, *Symplectic integrators for Hamiltonian problems: an overview*, Acta Numerica **1** (1991) 243–286.



I would like to conclude

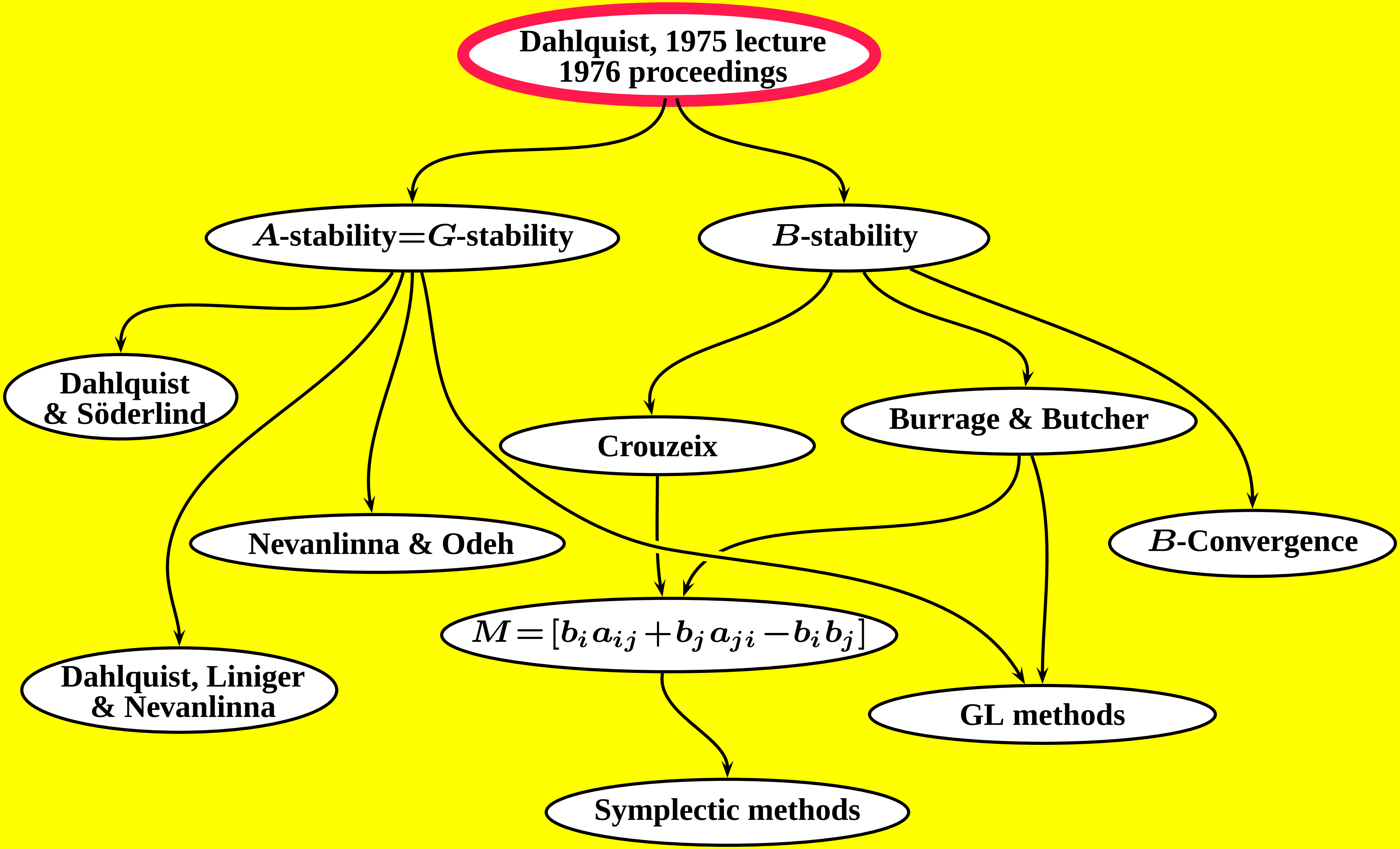
by recalling

the main ideas

from

Dahlquist's 1975 Dundee paper

G-stability and consequences



One-leg methods

Given a linear multistep method

$$\sum_{i=0}^{k} \alpha_i \widehat{y}_{n+i} = h \sum_{i=0}^{k} \beta_i f(\widehat{x}_{n+i}, \widehat{y}_{n+i}), \qquad \textbf{(*)}$$

normalized so that $\sum_{i=0}^{k} \beta_i = 1$,

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normalized so that $\sum_{i=0}^{k} \beta_i = 1$, define the "one-leg" counterpart as the method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = hf\left(x_{n+i}, \sum_{i=0}^{k} \beta_i y_{n+i}\right). \qquad \textbf{(**)}$$

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normalized so that $\sum_{i=0}^{k} \beta_i = 1$, define the "one-leg" counterpart as the method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h f\left(x_{n+i}, \sum_{i=0}^{k} \beta_i y_{n+i}\right). \qquad \textbf{(**)}$$

Note that the two methods have the same linear stability:

$$\rho(w) - z\sigma(w) = 0.$$

The two methods are very closely related.

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If y is a sequence computed using (**), and $\widehat{x}$ $\widehat{y}$ are defined by

$$\widehat{x}_n = \sum_{i=0}^{k} \beta_i x_{n+i}, \quad \widehat{y}_n = \sum_{i=0}^{k} \beta_i y_{n+i},$$

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Dahlquist's main aim is now to analyse the performance of a one-leg method with a dissipative nonlinear problem.

Nonlinear test problem

Dahlquist proposed use of the test problem

$y'(x) = f(x,y)$, where $\langle y - z, f(x,y) - f(x,z)\rangle \le 0$. **(*)**

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For convenience we will replace (*) by the related problem

$$y'(x) = f(x,y), \quad \text{where} \quad \langle y, f(x,y)\rangle \le 0.$$

The G norm

Let G be a positive-definite symmetric $k \times k$ matrix:

$$G = \begin{bmatrix} g_{00} & g_{01} & g_{02} & \cdots & g_{0,k-1} \\ g_{10} & g_{11} & g_{12} & \cdots & g_{1,k-1} \\ g_{20} & g_{21} & g_{22} & \cdots & g_{2,k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ g_{k-1,0} & g_{k-1,1} & g_{k-1,2} & \cdots & g_{k-1,k-1} \end{bmatrix}$$

and define the norm $\|\cdot\|_G$ by

$$\|Y_n\|_G^2 = \sum_{i,j=0}^{k-1} g_{ij} \langle y_{n+i}, y_{n+j} \rangle,$$

where $Y_n = [y_n, y_{n+1}, \ldots, y_{n+k-1}]$.

The contractivity property

For convenience, we will write $g_{ij} = 0$ if either i or j is outside the set $\{0, 1, 2, \ldots, k-1\}$.

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For convenience, we will write $g_{ij} = 0$ if either i or j is outside the set $\{0, 1, 2, \dots, k-1\}$.

A one-leg method (and the corresponding linear multistep method) is "G-stable" if G exists so that the symmetric part of the $(k+1) \times (k+1)$ matrix with elements

$$\alpha_i \beta_j + g_{ij} - g_{i-1,j-1}, \quad i, j \in \{0, 1, 2, \dots, k\},$$

is positive semi-definite.

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is positive semi-definite.

To appreciate the consequence of this definition, multiply by $\langle y_{n+i}, y_{n+j} \rangle$ and sum for $i, j = 0, 1, \ldots, k$.

Because of positive semi-definiteness, we have

$$T_1 + T_2 + T_3 \geq 0$$

where

$$\begin{aligned} T_1 &= \Big\langle \sum_{i=0}^{k} \alpha_i y_{n+i}, \sum_{j=0}^{k} \beta_j y_{n+j} \Big\rangle \\ &= h \Big\langle f\Big(\sum_{i=0}^{k} \beta_i y_{n+i}\Big), \sum_{j=0}^{k} \beta_j y_{n+j} \Big\rangle \leq 0 \\ T_2 &= \|Y_n\|_G^2 \\ T_3 &= -\|Y_{n+1}\|_G^2 \end{aligned}$$

implying that

$$\|Y_{n+1}\|_G \leq \|Y_n\|_G.$$

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Finally, I express my thanks to my colleagues in Auckland for advice and insight during the preparation of these notes.